HAKE

An Introduction to

Alternating Current Theory

Electrical Engineer

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AN INTRODUCTION TO ALTERNATING CURRENT THEORY

BY

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I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

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NOTATION

b	Susceptance.
c	Capacity.
E_m	Maximum value of e. m. f.
E	Effective value of e. m. f.
e	Instantaneous value of e. m. f.
f	Frequency in cycles per second.
g	Conductance
I_m	Maximum value of current.
I	Effective value of current.
i	Instantaneous value of current. Also energy component of current.
i'	Quadrature component of current.
j	$\sqrt{-1}$.
L	Inductance.
ℓ	Length.
M.M.F.	Magneto motive force.
μ	Magnetic permeability.
N	Number of turns of wire in a coil.
Φ	Maximum value of magnetic flux.
ϕ	Instantaneous value of magnetic flux.
P.F.	Power factor.
Q q	Quantity of electric charge.
R	Resistance.
θ	Angle of phase difference between current and e.m.f.

x_L	Inductive reactance.
x_c	Capacity reactance.
Y	Admittance
Z	Impedance of a circuit = $\sqrt{r^2 + x^2}$

AN INTRODUCTION TO ALTERNATING CURRENT THEORY

INTRODUCTION

In the study of alternating current theory there are certain fundamental principles which enter again and again into the problems and discussions. While these principles are in general quite simple, a complete knowledge and understanding of them is essential to a successful study of alternating current theory.

In the majority of texts many of these seemingly simple relations are assumed as previously known by the student. Since an error in this assumption often proves fatal to the students ultimate success, it seems worth while to present these facts in the simplest and most complete manner possible.

It is the purpose of this thesis to present a course, introductory to alternating currents, which, when thoroughly mastered, will give a reliable ground work for the study of more advanced theory and phenomena.

This course may be divided into seven general divisions as follows.

I. The generation of e. m. f. under various conditions of field flux and armature conductor grouping.

II. Effective values, waves and fundamental units.

III. Vectors.

IV. Power.

V. Effects due to resistance, inductive reactance, capacity reactance and calculations of circuits.

VI. Complex Quantities.

VII. Transmission lines.

The final division is merely a special case of circuits.

While these subdivisions may, and do, in many cases overlap they are in a sense separate, although the latter cases usually require an understanding of those preceeding.

I

GENERATION OF E. M. F.

Uniform Magnetic Field

Consider the e. m. f. generated by a single conductor moving across a magnetic field in a direction at right angles to the magnetic lines. This motion produces in the conductor an e. m. f. proportional to the rate at which the magnetic lines are cut, or the e. m. f. = $-\frac{d\phi}{dt}$ in the case of a single conductor, or if there are N conductors, the e. m. f. is $E = -N\frac{d\phi}{dt}$.

If the terminals of this conductor are connected by an electrical circuit a current will be set up. According to Lenz law the current set up by an e. m. f. acts in such a direction that its effect opposes the motion which produces the e. m. f. For this reason the factor $\frac{d\phi}{dt}$ should be prefixed by the negative sign as shown.

With a uniform distribution between pole pieces as shown in figure 1, if the conductor is moved with uniform velocity across the field in a direction perpendicular to the magnetic lines the e. m. f. generated is of constant value while the conductor is cutting the field. Since the conductor is cutting no lines the instant before it enters the field the e. m. f. generated is zero at that time but immediately upon entering the field the lines are cut at a certain

rate which remains constant throughout. The wave of e. m. f. may then be represented as rectangular in form and of constant intensity as long as the conductor is in the field. Figure 1 shows also the e. m. f. generated by the conductor in moving from outside the field directly across it to a point on the opposite side. The second part of the wave, that is the negative loop, is generated by the conductor when moving across the field in a direction opposite to that of the original motion.

Consider now that a coil of wire AA, figure 1, has uniform rotary motion about the axis o. If this motion is uniform the coil will sweep over equal portions of the arc during all equal increments of time dt. During these equal intervals of time the coil will not cut equal numbers of magnetic lines. At the time it is in the vertical position it moves parallel to the field while, when in the horizontal position, it moves perpendicular to the field.

Thus the e. m. f. is a minimum when the coil is in the vertical position and a maximum when the coil is in the horizontal position. It is seen from the figure that when the coil is horizontal the number of lines enclosed by it is zero and the number of enclosed lines is a maximum when the coil is in the vertical position.

In figure 1 the number of magnetic lines is taken as 40 and increments of time chosen for every 10°. The data thus obtained is tabulated in Table I.

Table I

Enclosed Flux and E. M. F. Generated by Coil in Fig. 1.

Time	ϕ	$d\phi$	e	$\angle\theta$
0	40			
		0	0	5
10	40			
		-.5	.07	15
20	39.5			
		-3.	.4	25
30	36.5			
		-4.	.55	35
40	32.5			
		-5.	.69	45
50	27.5			
		-6.5	.89	55
60	21.0			
		-6.5	.96	65
70	14.5			
		-7.2	.99	75
80	7.3			
		-7.3	1.	85
90	0			
		-7.3	1.	95
100	-7.3			
		-7.2	.99	105
110	-14.5			
		-6.5	.96	115
120	-21.0			
		-6.5	.89	125
130	-27.5			
		-5.	.69	135
140	-32.5			
		-4.	.55	145
150	-36.5			
		-3.	.4	155
160	-39.5			
		-.5	.07	165
170	-40			
		0	0	175
180	-40			
			0	185

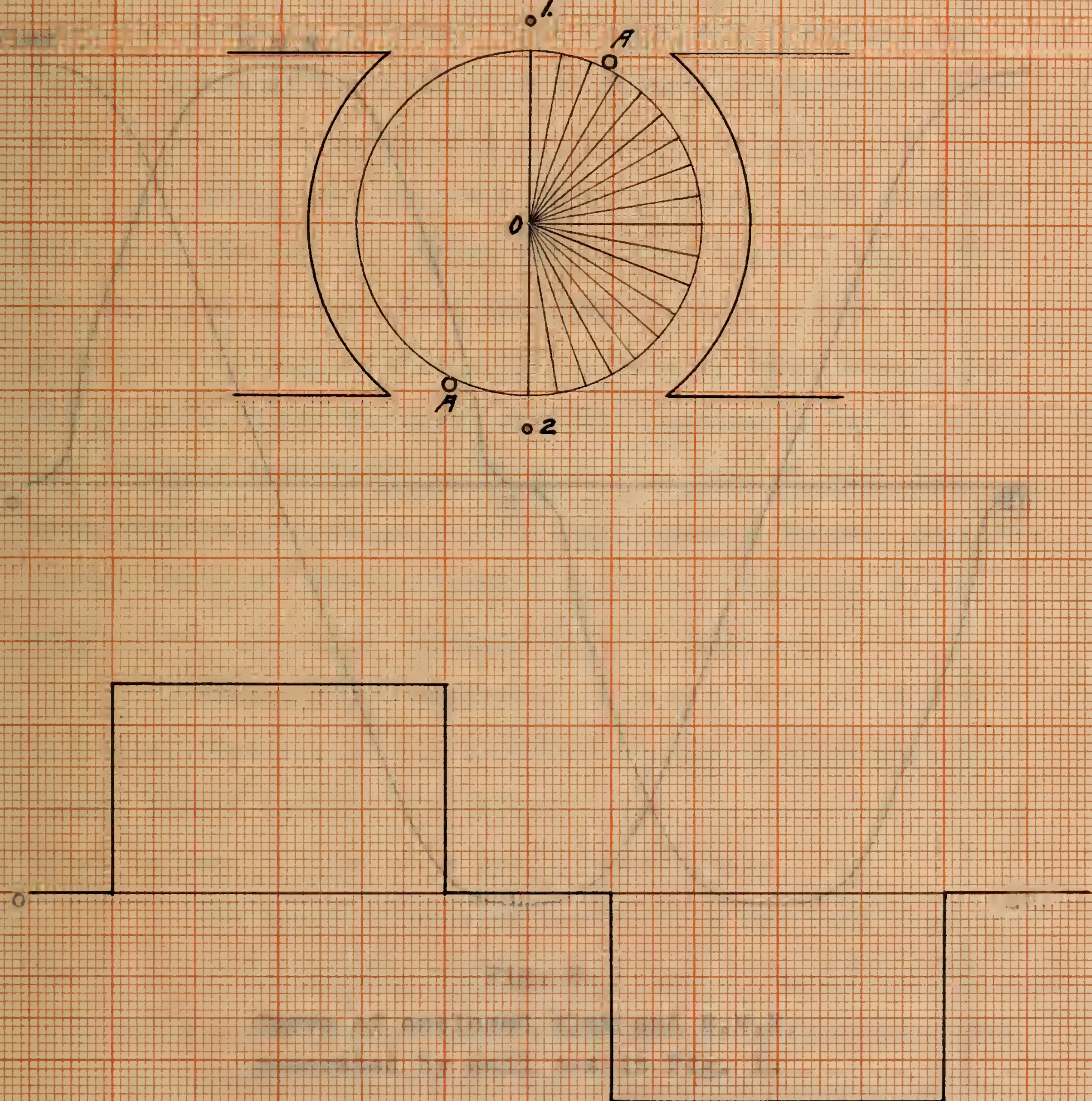
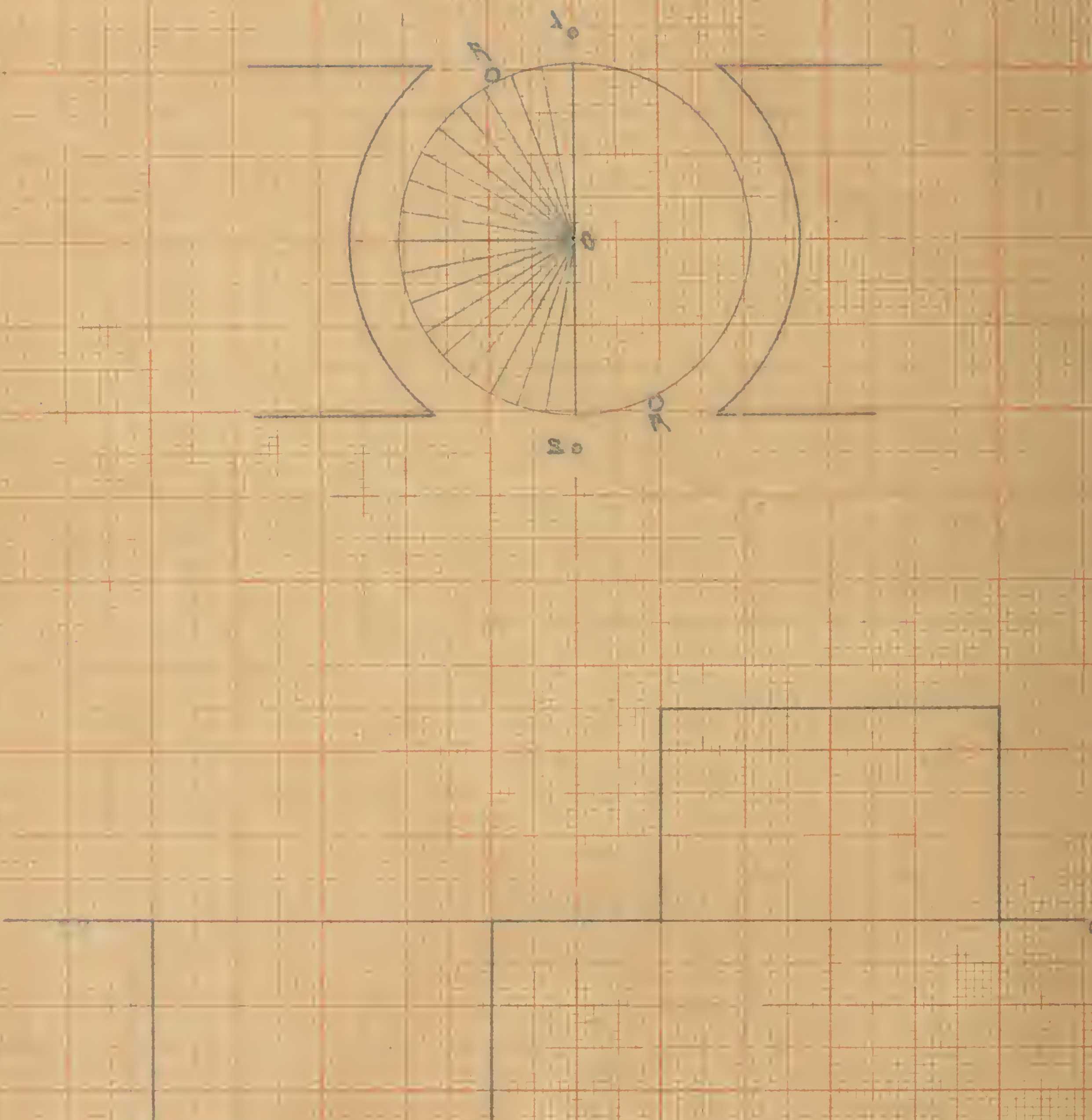


Fig. 1

Uniform magnetic field and E.M.F.
generated by a conductor moving vertically
from 1 to 2 and back.
No iron between the poles.



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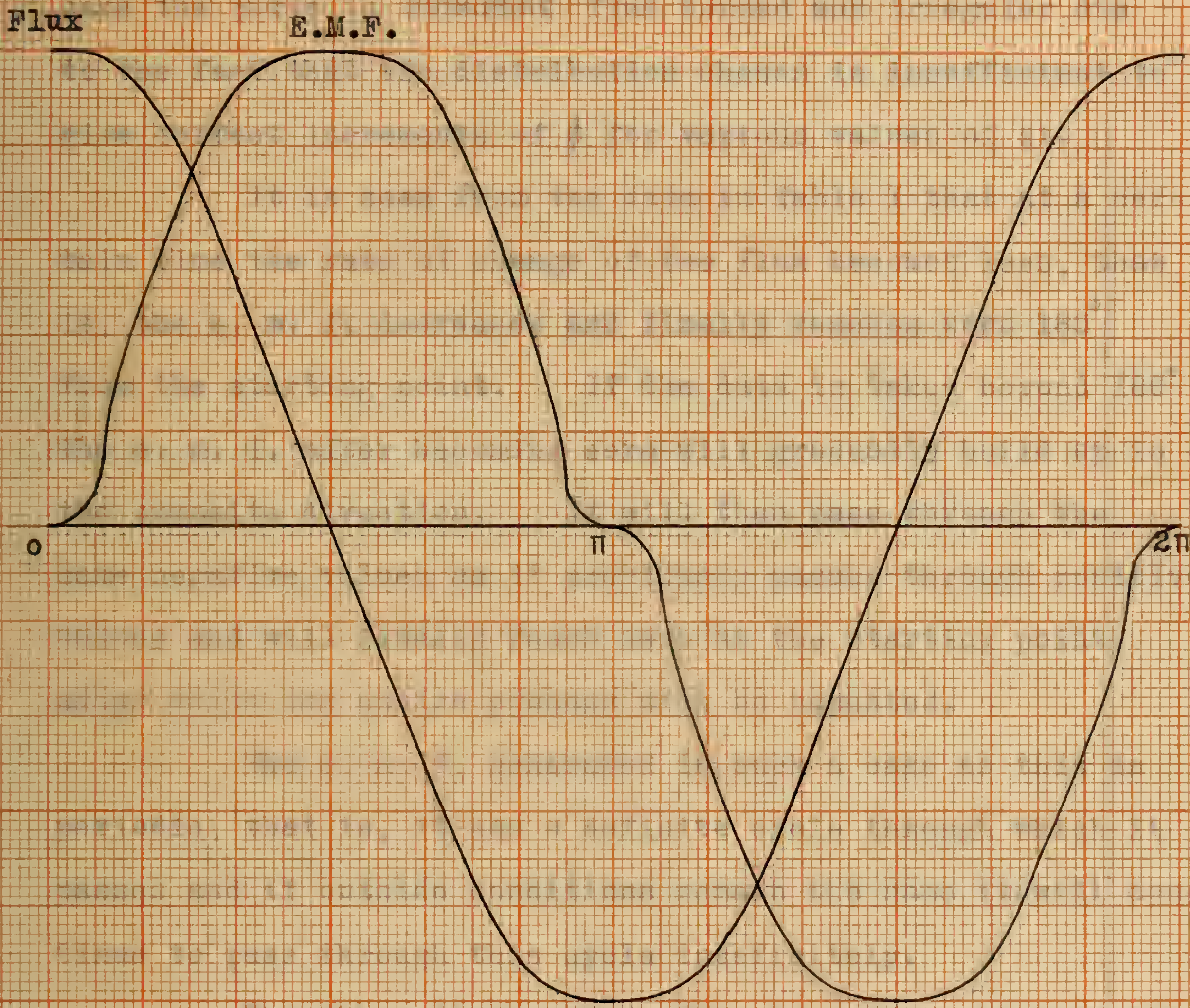


Fig. 2

Curve of enclosed flux and E.M.F. generated by coil A-A in Fig. 1.

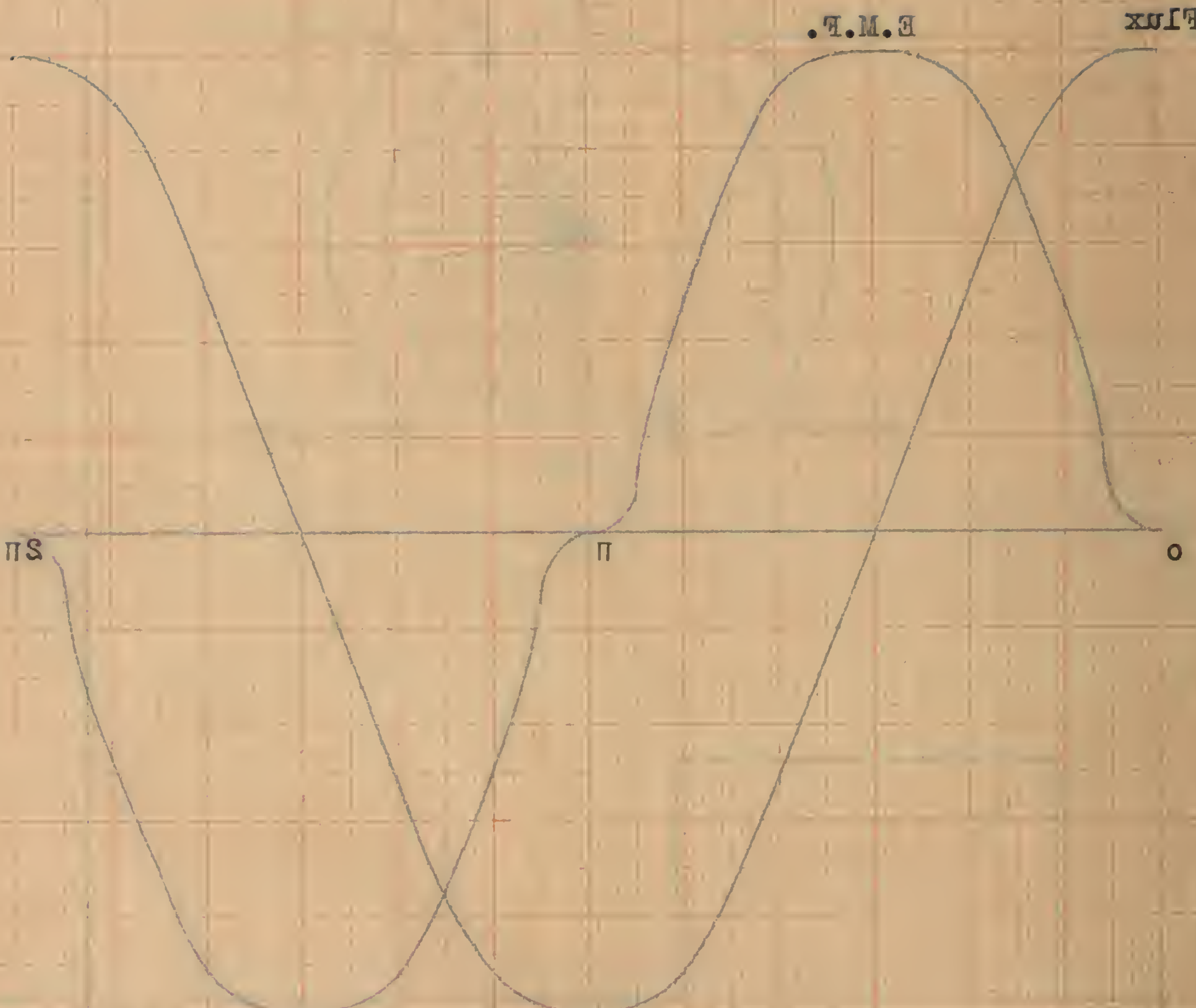


Fig. 2

Curve of enclosed flux and E.M.F.
generated by coil A-A in Fig. 1.

The relation between time and e. m. f. when plotted to rectangular coordinates as in figure 2 takes a form similar to that of a sine curve. In this particular case the curve is somewhat flat topped and irregular due to the fact that the distribution chosen is insufficient to give correct increments of ϕ for various values of dt.

It is seen from the data in Table I that at a certain time the rate of change of the flux becomes less, that is, the e. m. f. decreases and finally reaches zero 180° from the starting point. If the data is taken beyond 180° the e. m. f. after becoming zero will gradually build up in the opposite direction. It will then pass through the same negative values as it previously passed through positive values and will finally reach zero at the starting point, after which the entire process will be repeated.

The e. m. f. generated in such a case as this is periodic, that is, it has a definite cycle through which it passes and if outside conditions remain the same it will continue to pass through this cycle indefinitely.

The above discussion refers to a two pole magnetic field. It is readily seen however that it would hold equally well for any even number of field poles. In any case the e. m. f. generated passes through one complete cycle per revolution for each pair of field poles. Then in a two pole machine the e. m. f. passes through one cycle per revolution, in a four pole machine through two cycles per revolution, in a six pole machine through three cycles per revolution etc.

Non-Uniform Magnetic Field

Consider in this connection the case in which the field is concentrated at the pole tips as in figure 3. Table II gives the relation between coil position and e. m. f. generated for the case shown in figure 3, in which the total number of lines is taken as 38. The data in this Table when plotted to rectangular coordinates gives the curve in figure 3. This curve is of course considerably more irregular than would be found in practice with a similar distribution of flux due to the fact that the distribution here chosen is such that correct increments can hardly be obtained. In an actual machine such a distribution as this would give a wave with two peaks, similar to the one shown, but the changes in magnitude would not be so abrupt.

Table II

Enclosed Flux and E. M. F. Generated by a

Coil in a Field as shown in Fig. 3.

Angle	ϕ	$d\phi$	For	e
0	38			
		0	5	0
10	38			
		-4	15	.66
20	34			
		-6	25	1.
30	28			
		-6	35	1.
40	22			
		-6	45	1.
50	16			
		-4	55	.66
60	12			
		-4	65	.66
70	8			
		-4	75	.66
80	4			
		-4	85	.66
90	0			
		-4	95	.66
100	-4			
		-4	105	.66
110	-8			
		-4	115	.66
120	-12			
		-4	125	.66
130	-16			
		-6	135	1.
140	-22			
		-6	145	1.
150	-28			
		-6	155	1.
160	-34			
		-4	165	.66
170	-38			
		-0	175	0
180	-38			
		-0	185	0

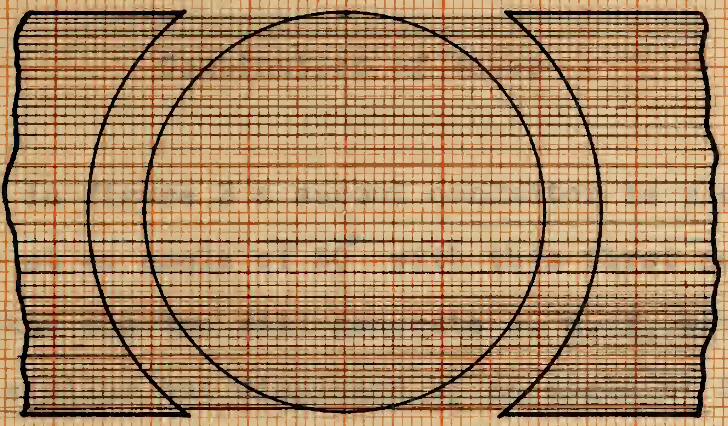
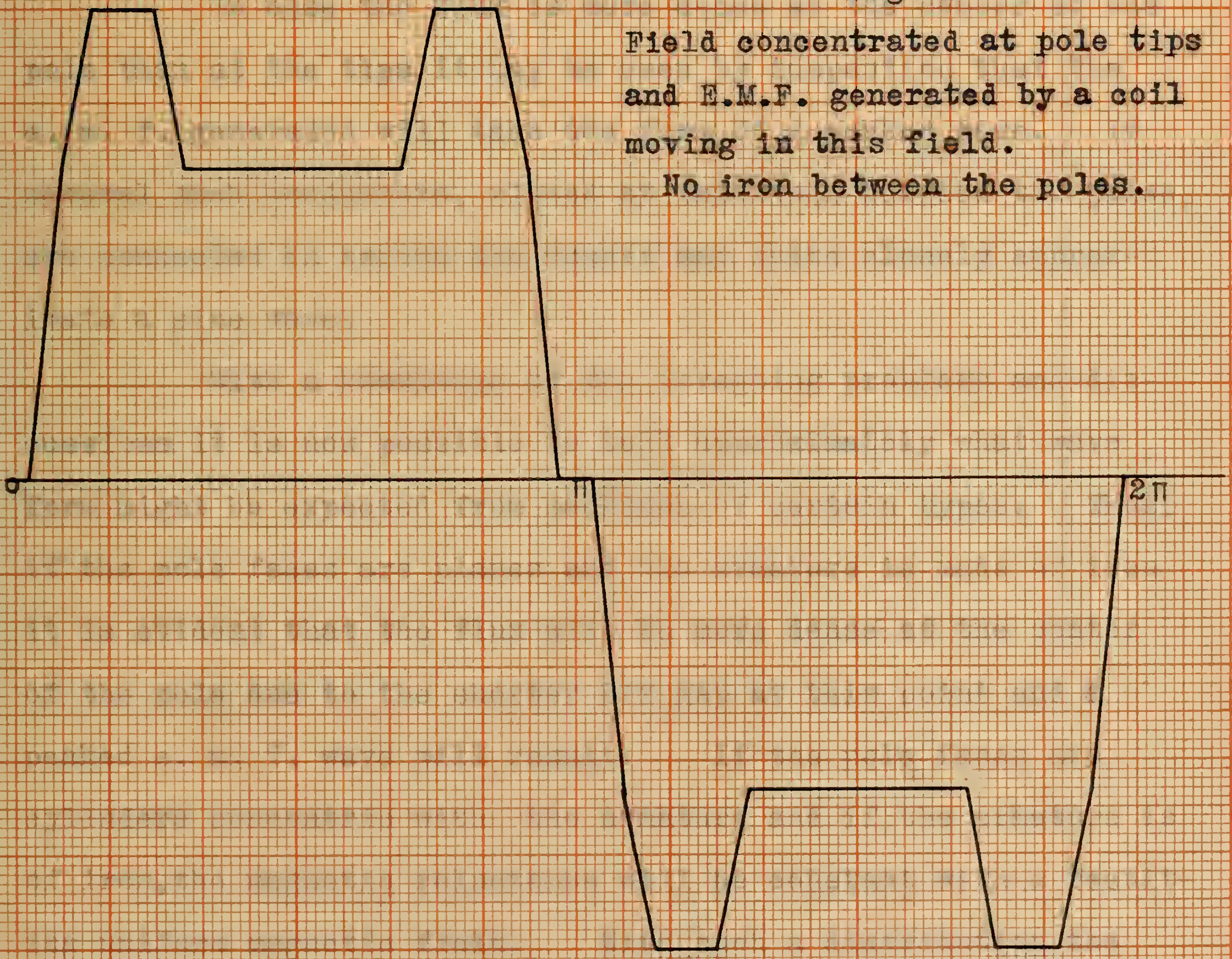


Fig. 3

Field concentrated at pole tips and E.M.F. generated by a coil moving in this field.

No iron between the poles.



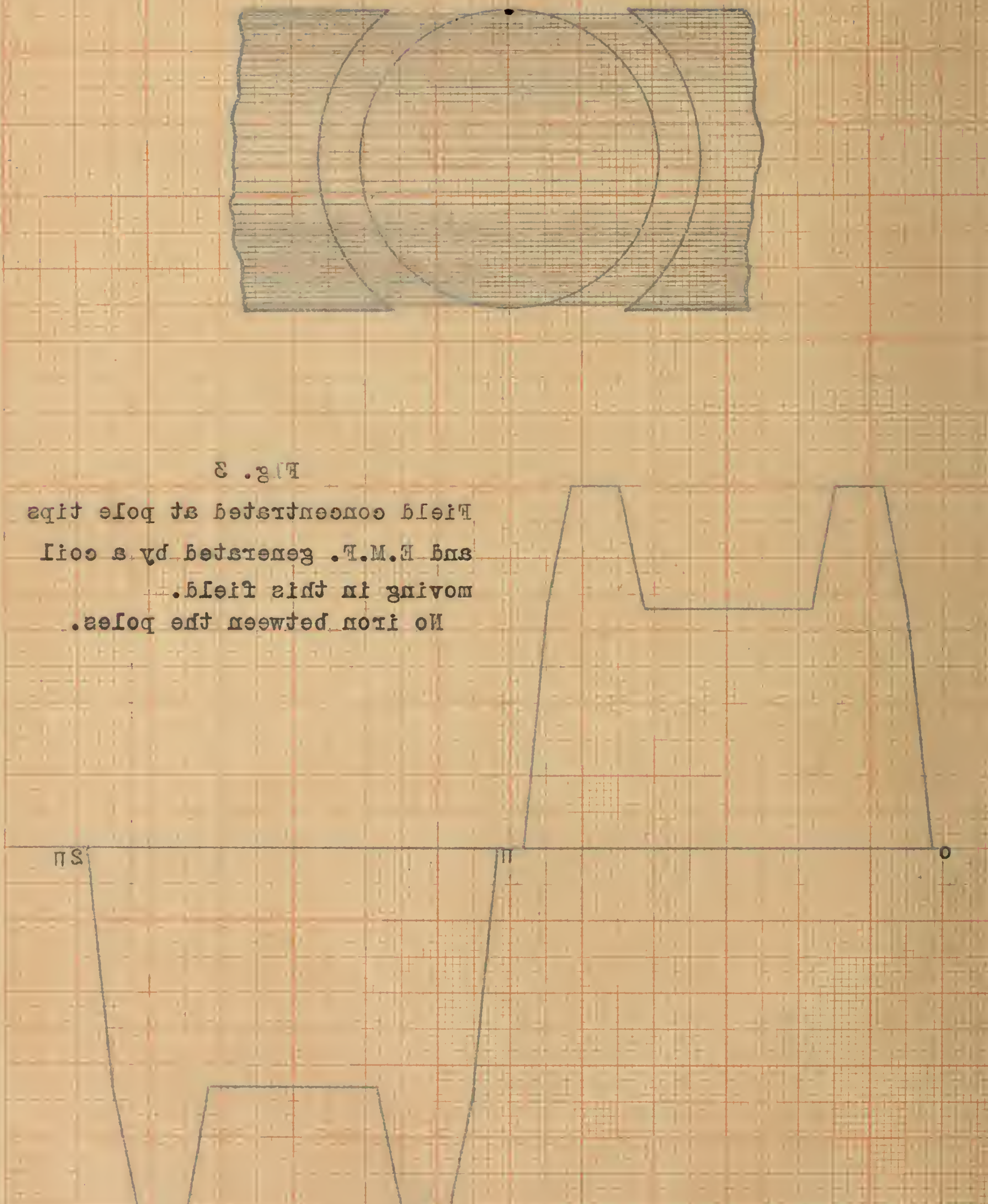


Fig. 3

Field concentrated at pole tips
and E.M.F. generated by a coil
moving in this field.
No iron between the poles.

Combination of Waves

If in figure 3 a second conductor is placed on the armature at an angle of 30° with this first conductor it is evident that the two will generate e. m. f. waves which are exactly similar but displaced by an angle of 30° , see A and B figure 4. If these two conductors are connected in series their e. m. f.'s will be added together and the resultant wave C, figure 4 will be produced.

In case the flux is more dense at the center of the pole than at the tips it may be seen by inspection that the e. m. f. generated will take the form of a peaked wave. If several such conductors, placed at intervals between the poles, are connected in series the result may quite closely approximate a sine wave.

With a knowledge of the foregoing problems and discussions it is now possible to tell approximately what wave form might be expected from machines of certain types. Thus, if the pole faces are planes and the armature is made of iron it is evident that the flux will be more dense at the center of the pole due to the shorter air gap at this point and a peaked e. m. f. wave will result. If the pole faces are cylinders concentric with the armature and if the armature is of iron, the magnetic reluctance will be constant with a resulting uniform magnetic field. With such a distribution the e. m. f. generated is rectangular in form since the flux cut during any given increment of time is the same as for all other

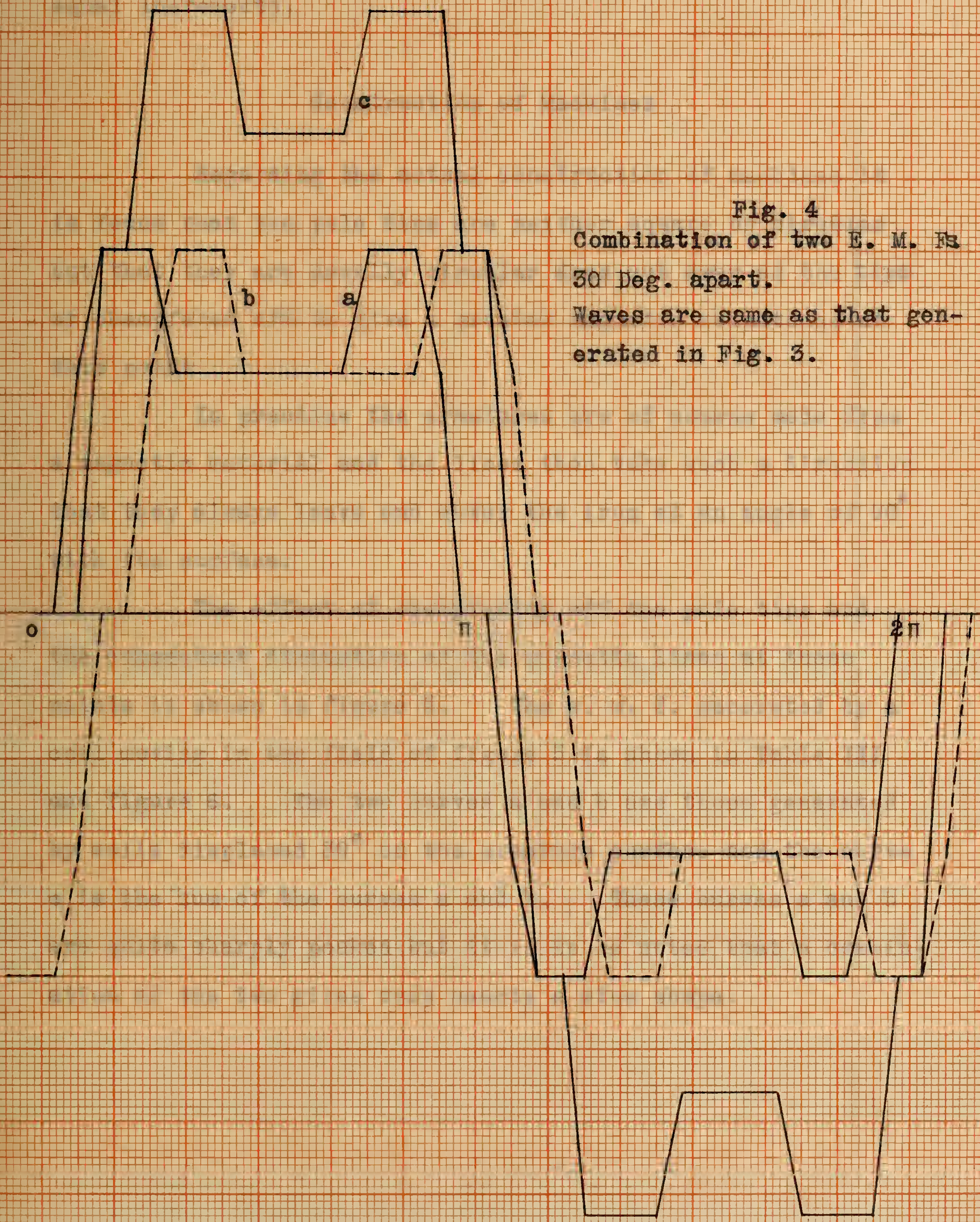


Fig. 4
Combination of two E. M. Fs 30 Deg. apart.
Waves are same as that generated in Fig. 3.

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Combination of two E. M. F.
30 Deg. apart.
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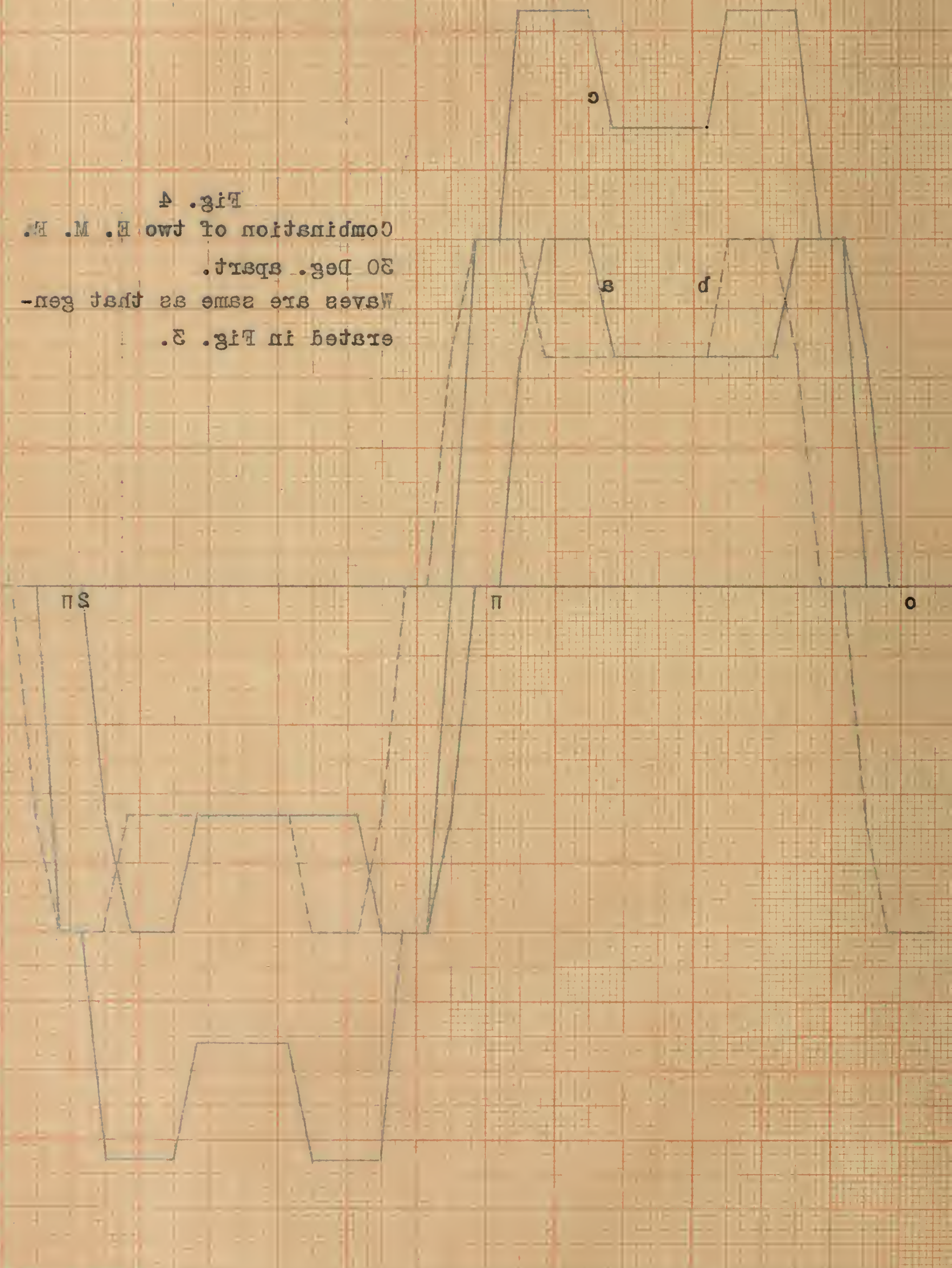


equal increments.

Construction of Machines

Regarding the actual construction of machines it is found that the pole tips are neither square or circular but that they are usually circular arcs cut away at the tips or champfered off to give a greater magnetic reluctance at this point

In practice the armatures are of course made from a magnetic material and the lines then take such a direction that they always leave and enter the iron at an angle of 90° with its surface.

The effect of champfering off the pole tips and the consequent diminution of the magnetic lines at these points is shown in figure 5. The e. m. f. generated by a coil moving in the field of figure 5 is shown in Table III and figure 6. The two curves a and b are those generated by coils displaced 30° on the armature surface and the curve c is the sum of the curves a and b. These curves a and b are quite sharply peaked but it is to be noted that a combination of the two gives very nearly a sine shape.

Table III

Enclosed Flux and E. M. F. Generated by Coil in Fig. 5.

Angle	ϕ	$d\phi$	e	For deg.
0	34			
		-0	0	5
10	34			
		-0	0	15
20	34			
		-1	.125	25
30	33			
		-3	.375	35
40	30			
		-4	.5	45
50	26			
		-5	.125	55
60	21			
		-6	.75	65
70	15			
		-7	.875	75
80	8			
		-8	1.	85
90	0			
		-8	1.	95
100	8			
		-7	.875	105
110	15			
		-6	.75	115
120	21			
		-5	.625	125
130	26			
		-4	.5	135
140	30			
		-3	.375	145
150	33			
		-1	.125	155
160	34			
		-0	0	165
170	34			
		-0	0	175
180	34			

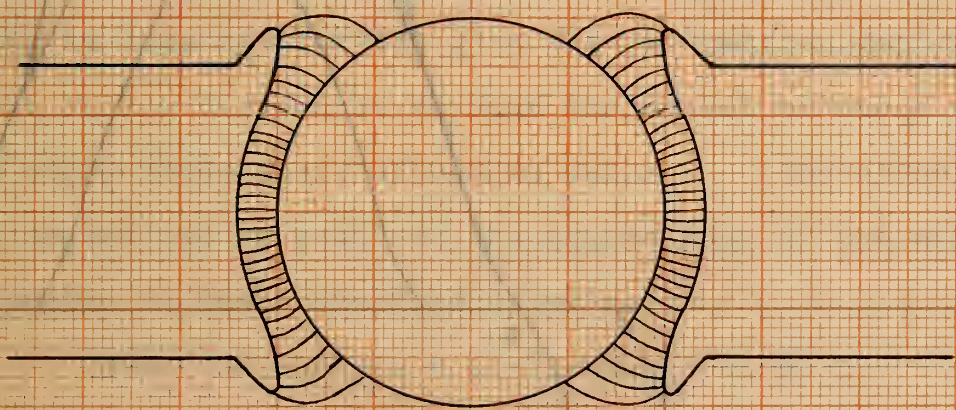
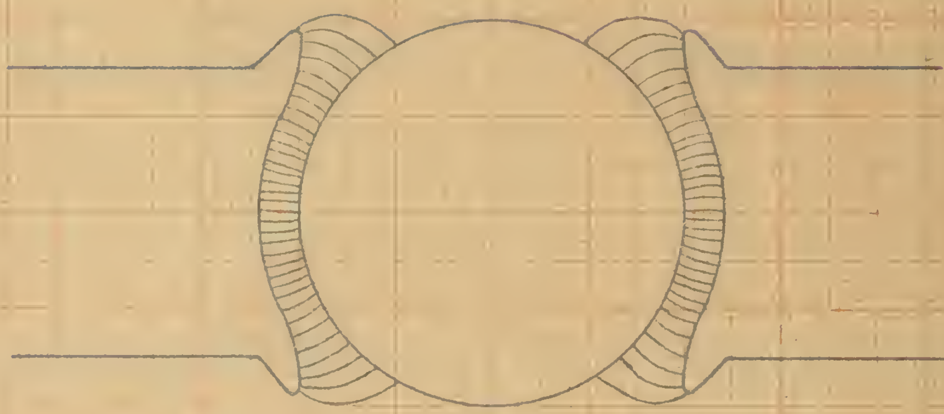


Fig. 5

Magnetic field between pole tips.
Armature made of magnetic material.



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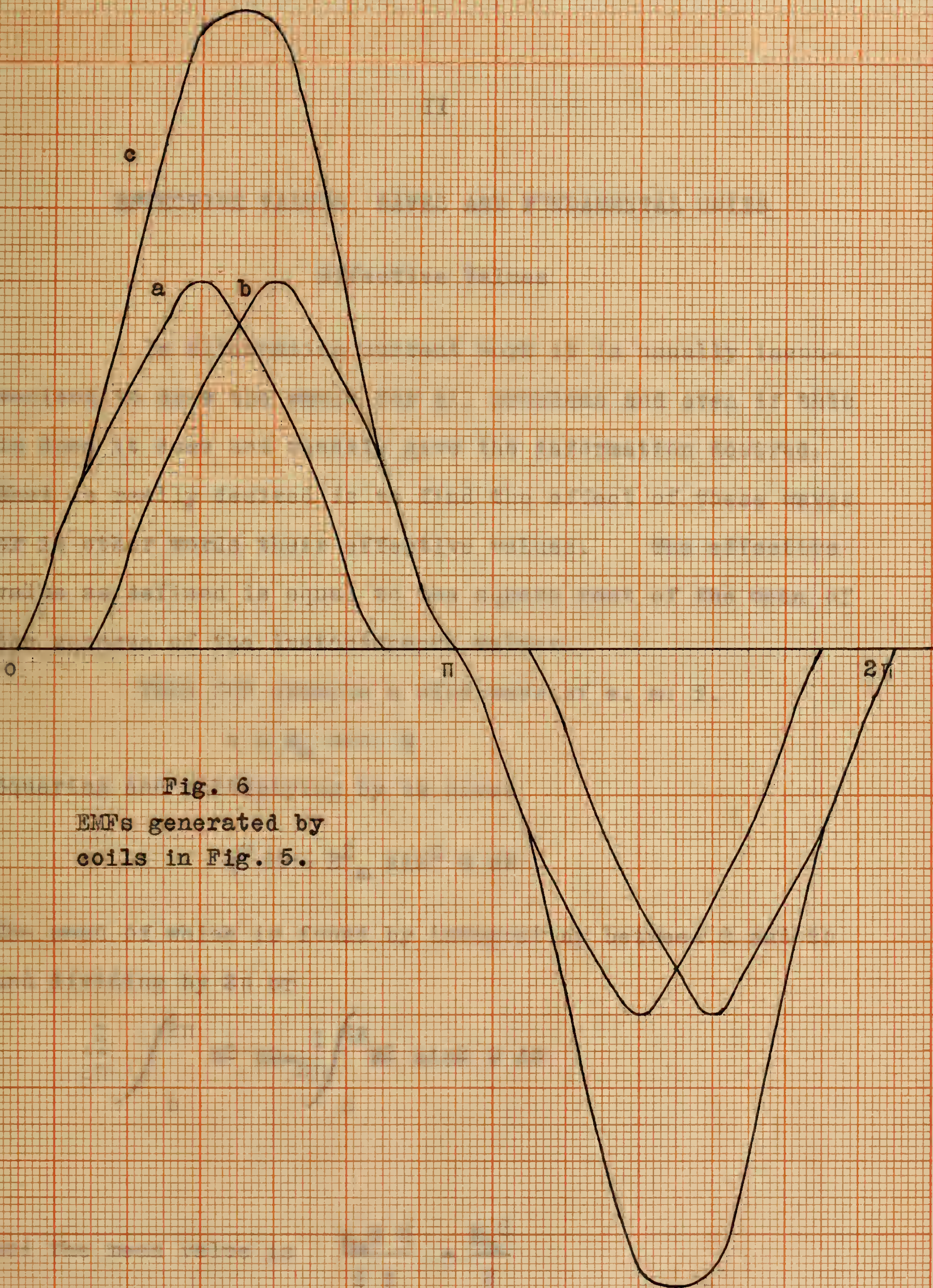


Fig. 6
EMFs generated by
coils in Fig. 5.

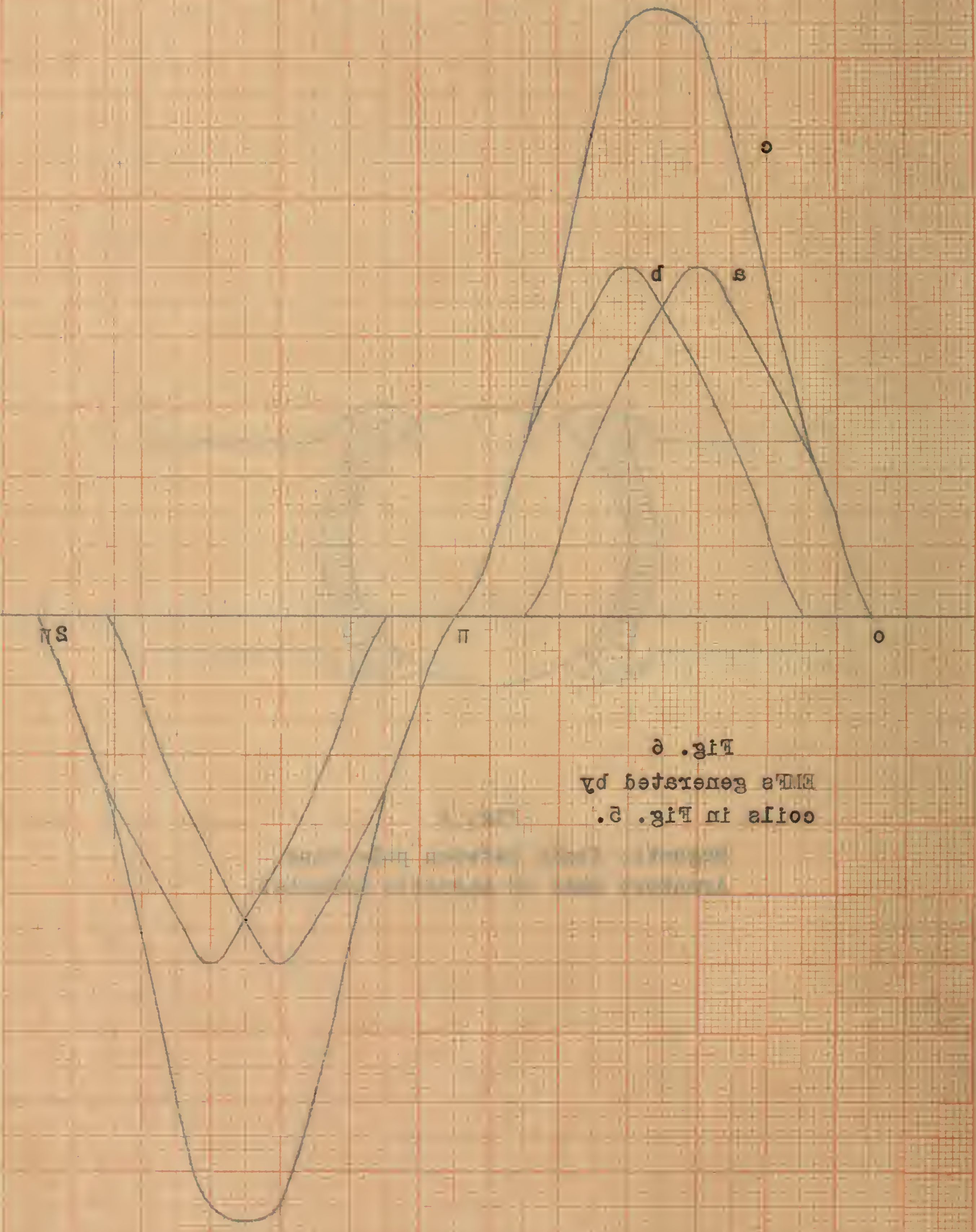


Fig. 6
EMFs generated by
coils in Fig. 5.

II

EFFECTIVE VALUES, WAVES AND FUNDAMENTAL UNITS

Effective Values

In alternating current work it is usually inconvenient to draw the waves for all problems and even if this is done it does not readily give the information desired. What is really desired is to find the effect of these waves or in other words their effective values. The effective value as defined is equal to the square root of the mean of the squares of the instantaneous values.

Take for example a sine wave of e. m. f.

$$e = E_m \sin \theta$$

Squaring and multiplying by $d\theta$ gives

$$e^2 \, d\theta = E^2_m \sin^2 \theta \, d\theta$$

The mean of which is found by integrating between 0 and 2π and dividing by 2π or

$$\frac{1}{2\pi} \int_0^{2\pi} e^2 \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} E^2 \sin^2 \theta \, d\theta$$

and the mean value is $\dfrac{E_m^2 \pi}{2\pi} = \dfrac{E_m^2}{2}$

From the definition the effective value is the square root of the mean of the squares, then $E = \frac{E_m}{\sqrt{2}}$, or the effective value of a sine wave of e. m. f. or current is equal to the maximum value divided by $\sqrt{2}$.

Average Value

The average value of a wave is also frequently used and is, as its name implies an average of the ordinates of the curve. The half cycle must be chosen in this case for if the average is considered for 360 degrees it will be zero if the curve is symmetrical.

If $e = E_m \sin \theta$ is the equation of an e. m. f. wave, to find the average value it is necessary to integrate the curve between o and π. The area thus found when divided by the base gives the average value.

$$\frac{1}{\pi}\int_0^\pi e\, d\theta = \frac{1}{\pi}\int_0^\pi E_m \sin\theta\, d\theta$$

$$\frac{1}{\pi}\int_0^\pi e\, d\theta = -\frac{E_m \cos\theta}{\pi}\Big]_0^\pi$$

$$= -\frac{E_m}{\pi}(-1 -1) = \frac{2E_m}{\pi}$$

And the average is $\frac{2\,E_m}{\pi} = .637\,E_m$. That is, the average value of a sine wave of e. m. f. or current is .637 times the maximum.

Since the effective value of e. m. f. or current is

a desirable quantity it is well to know how it may be found without first plotting out the wave and then obtaining the effective value by graphical methods or by mathematical ones provided the equation of the curve is known.

Expression for E. M. F.

Consider a coil having N turns interlinked by a flux Φ. During the first quarter cycle while the flux changes from zero to a maximum there are $N \Phi$ interlinkages of flux and similarly for the next quarter cycles, or

During first 1/4 cycle there are $N \Phi$ interlinkages.
During second 1/4 cycle there are $N \Phi$ interlinkages.
During third 1/4 cycle there are $N \Phi$ interlinkages.
During fourth 1/4 cycle there are $N \Phi$ interlinkages.

Then for one complete cycle there are $4N\Phi$ average flux interlinkages of the coil. If there are f cycles per second the average interlinkages per second will be $4N\Phi f$, which if multiplies by 10^{-8} gives the average value of the voltage in practical units induced in the coil or

$$e_{av} = 4 N \Phi f 10^{-8}$$

but $e_{av} = .637 E_m$ and $E = .707 E_m$, therefore

$$\frac{e_{av}}{.637} = \frac{E}{.707}$$

or it may be written

$$e_{av} = \frac{2 E_m}{\pi}$$

$$E = \frac{E_m}{\sqrt{2}}$$

from which

$$e_{av} = \frac{2\sqrt{2}}{\pi} E$$

this when substituted in the above equation gives

$$E = \frac{4 \pi N \Phi f 10^{-8}}{2 \sqrt{2}}$$

$$= 4.44 \Phi f N 10^{-8}$$

The above expression for voltage may also be developed as follows. Assume a sine wave of flux

$$\phi = \Phi \cos \omega t \qquad \text{and}$$

$$e = -N \frac{d\phi}{dt}$$

$$\frac{d\phi}{dt} = -\Phi \omega \sin \omega t$$

$$e = \omega \Phi N \sin \omega t$$

When $\sin \omega t$ is a maximum $e = E_m$, or

$$E_m = \omega \Phi N$$

$$= 2 \pi f \Phi N$$

$$E = \frac{E_m}{\sqrt{2}} = 4.44 \Phi f N \quad \text{ab. volts}$$

$$= 4.44 \Phi f N 10^{-8} \quad \text{volts.}$$

E. M. F. Consumed by Resistance

Assume a sine wave of e. m. f. impressed upon a

circuit of resistance r. Starting at the time when the e. m. f. is zero and applying Ohms law it is seen that the current at this instant is zero. As the e. m. f. increases the current also increases. It has its maximum value at the same time as the e. m. f. and both are again zero at the same instant. Thus it is said that in a circuit having resistance only, the current and e. m. f. are in phase.

E. M. F. of Self Inductance

If a sine wave e. m. f. is impressed upon a coil of wire having an air core a sine wave of current will flow in the coil. From the preceding paragraph it is seen that the e. m. f. consumed by resistance is in phase with the current. This current sets up a magneto motive force which is at all times proportional to it and since the coil has an air core the magneto motive force produces a flux ϕ which is proportional to the current and in phase therewith as represented in figure 7. This flux ϕ when threading through the coil cuts the turns of the wire and induces therein an e. m. f. which is proportional to the rate of change of the flux and the number of turns in the coil. Since the induced e. m. f. is due to the rate of change of the flux and is opposed to the change that is taking place, that is $e = -N\frac{d\phi}{dt}$, it is the negative derivative curve of the flux and since the flux is a sine wave this curve will also be a sine wave but displaced $90°$ behind the flux curve as shown in figure 7. In order that the current may flow there must be impressed upon the coil an e. m. f. to overcome this self induced e. m. f. as

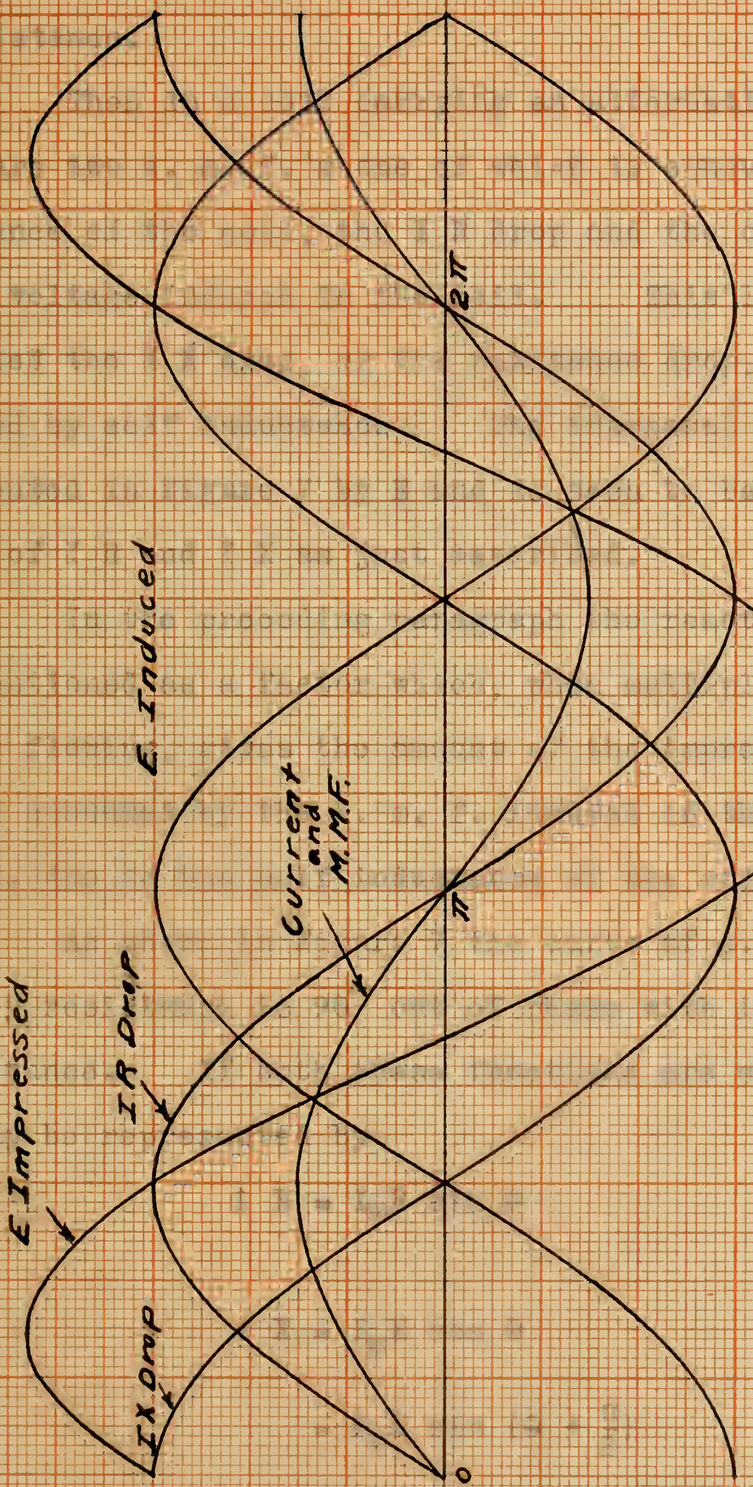


Fig 7
$I_m = 1.$ $R=2$ $X=2$

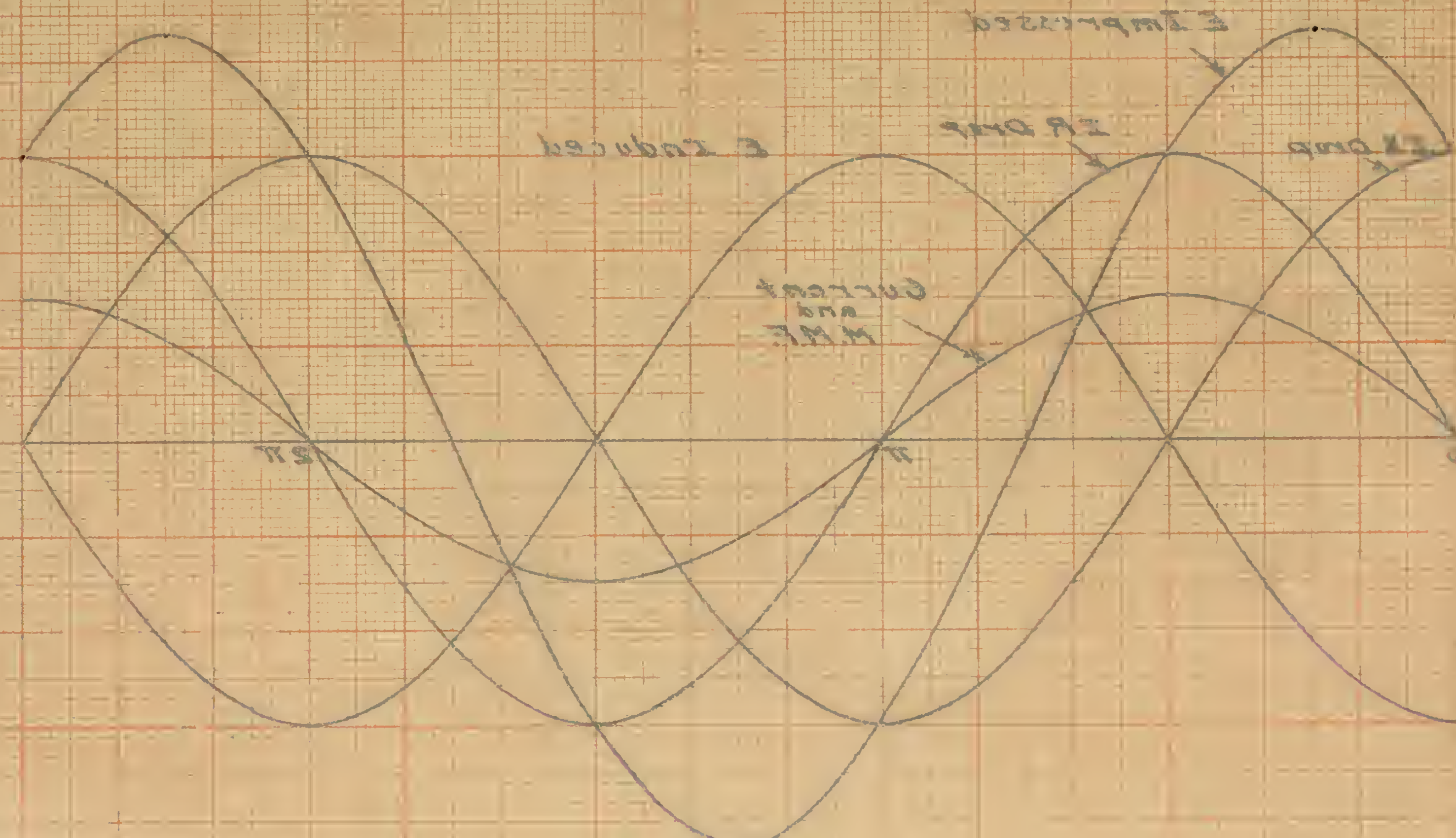


Fig 7
$I_m = 1 \quad R = 2 \quad X = 2$

well as that necessary to make up for the e. m. f. comsumed by resistance.

Then in a coil carrying an alternating current there are two e. m. f.'s one of which is consumed by the resistance of the coil, the I R drop and the other consumed by the voltage induced in the coil. This latter voltage is called the I X drop, or the reactance drop, or the e. m. f. consumed by self inductance. The impressed e. m. f. is represented in figure 7 by E and is seen to be the sum of curves of I R and I X as just described.

In the preceding paragraph the reactance X has been mentioned as a factor which, when multiplied by the current flowing, gives the amount of the impressed e. m. f. that is consumed by the e. m. f. induced in the coil or the e. m. f. due to the self inductance of the coil.

As shown in figure 7 the curve of e. m. f. consumed by resistance is 90° out of phase with that consumed by reactance. If both these functions are sine curves they may be represented by

$$i\,R = I_m R \sin \theta$$

and

$$i\,X = I_m X \cos \theta$$

$$= I_m X \sin \left(\theta + \frac{\pi}{2}\right)$$

which shows that the i X curve is 90° ahead of the i R curve and therefore the sum of the two is a sine curve generated by a radius vector forming the diagonal of the rectangle

of which I R is one side and I X the other. The value of this diagonal must be

$$E = \sqrt{\overline{I R}^2 + \overline{I X}^2}$$

but since $E = I Z$

$$Z = \sqrt{R^2 + X^2}$$

In which Z is the total impedance of the circuit.

Determination of X

Referring again to figure 7 it is seen that at any instant the impressed e. m. f. is equivalent to the sum of the instantaneous values of I r and I x. Since this I x drop is proportional to the change of flux in the coil and since this change of flux is proportional to the change in current and the maximum value of the flux is proportional to the number of turns in the coil the expression for e. m. f. may be written as $e = -L \frac{di}{dt}$ instead of $e = -N \frac{d\phi}{dt}$. The term L is the inductance factor and depends upon the number of turns of wire in the coil and the shape of the coil as shown later. Using the above expression for the e. m. f. of self inductance the equation for the voltage at any time may be written as

$$e = i r + L \frac{di}{dt}.$$

The positive sign is used here since the e. m. f. $L \frac{di}{dt}$ is that which must be impressed to overcome the e. m. f. of self inductance. If the value of the resistance is very small the above equation may be written $e = L \frac{di}{dt}$ and if the

current flowing is

$$i = I_m \sin \omega t$$

Then $\frac{di}{dt} = I_m \omega \cos \omega t$

which when substituted in the equation for e gives

$$e = L I_m \omega \cos \omega t$$

When $\cos \omega t = 1$, e is a maximum, that is $e = E_m$ and $E_m = \omega L I_m$. If these maximum values are replaced by effective values and the expression $2 \pi f$ is substituted for the angular velocity ω

$$E = 2 \pi f L I$$

The term $2 \pi f L$ is usually desègnated by the letter x in which case

$$E = x I.$$

This interpreted means that the impressed voltage is just equal to the product of the current times the reactance, which is true in case there is no resistance in the circuit.

The above expression for x may also be proved by assuming the e. m. f. formula previously deduced

$$E = 4.44 \Phi f N 10^{-8}$$

In a coil of one turn the effective value of the flux is $\phi_{eff} = L I$ and $\Phi = L I_m = L I \sqrt{2}$.

Substituting this in the equation for E gives

$$E = \sqrt{2}\, 4.44 \; L I f 10^{-8}$$

$$= 2 \pi f L I 10^{-8}$$

in which L is expressed in C.G.S. units. This when reduced to practical units gives

$$E = 2 \pi f L I = x I$$

in which the value of the reactance x is expressed in ohms and the inductance L in henrys.

Unit of Inductance

The henry is defined as follows: - If in a given circuit a change of current of one ampere per second generates one volt the circuit is said to have an inductance of one henry.

In a coil of wire having N turns and I amperes flowing the M. M. F. in gilberts is = .4 π N I. The reluctance is $\frac{\ell}{\mu A}$ from which $\phi = \frac{M.M.F.}{R} = \frac{.4 \pi N I \mu A}{\ell}$. At any time the current is stopped there are N ϕ interlinkages or interlinkages = $\frac{.4 \pi I N^2 \mu A}{\ell}$ and if the change in flux occurs in one second the e. m. f. generated is

$$E = \frac{.4 \pi N^2 I \mu A}{\ell} \quad \text{absolute volts}$$

and if the current is unity

$$E = L \text{ henrys} = \frac{.4 \pi N^2 \mu \; A \, 10^{-8}}{\ell}$$

Problem:- Given a coil of 200 turns wound about a ring having a mean circumference of 100 c.m. and a cross sectional area of 10 sq. c.m. Find the value of L and x for a frequency of 60 cycles. First for the coil having an air core and then for the coil having an iron core of permeability μ = 2000.

Solution:-

$$L = \frac{.4\ \pi\ N^2 \mu\ A}{l\ 10^8}$$

$$= \frac{.4\ \pi\ 200^2\ 1\ x\ 10}{100\ x\ 10^8} = .0000504 \text{ Henrys.}$$

$$x = 2\ \pi\ f\ L = 2\ \pi\ 60\ x .0000504 = .019 \text{ ohms}$$

for air in which $\mu = 1$

For $\mu = 2000$ $\quad$ $L = .1008$ $\quad$ $x = 38.$

Equivalent Sine Waves

In practice the waves of current and e. m. f. are not usually simple harmonic relations, that is they are not sine waves. In order that the theory as developed may be made to apply in these cases it is necessary to consider in the calculations that these distorted waves may be replaced by sine waves of the same effective values. This assumption is in general sufficiently accurate for practical purposes although there may be some conditions under which the results are considerably in error.

III

VECTORS

Thus far alternating currents and e. m. f.'s have been represented by sine waves plotted in such manner as to show the phase relations of the various quantities. There are instances however in which these plotted waves become somewhat confusing and further, the time required for this is very appreciable and it can not be accurately done except from tables or by means of standard curves. For these reasons it has become customary to represent the waves by rotating vectors and the diagrams representing the relations are known as vector diagrams.

Generation of Sine Curves by Rotating Vectors

That a harmonically varying function can be represented by a rotating vector may be shown as follows. Consider a line as in figure 8 rotating at uniform velocity about one fixed end. If the extremities of this line are projected upon a vertical line h which is moving at uniform velocity in a horizontal direction it will be found that the extremities of this projected line fall in a sine curve and since this is the case the varying function may be represented by the rotating vector rather than by a sine curve.

The rotating vector is a straight line drawn to

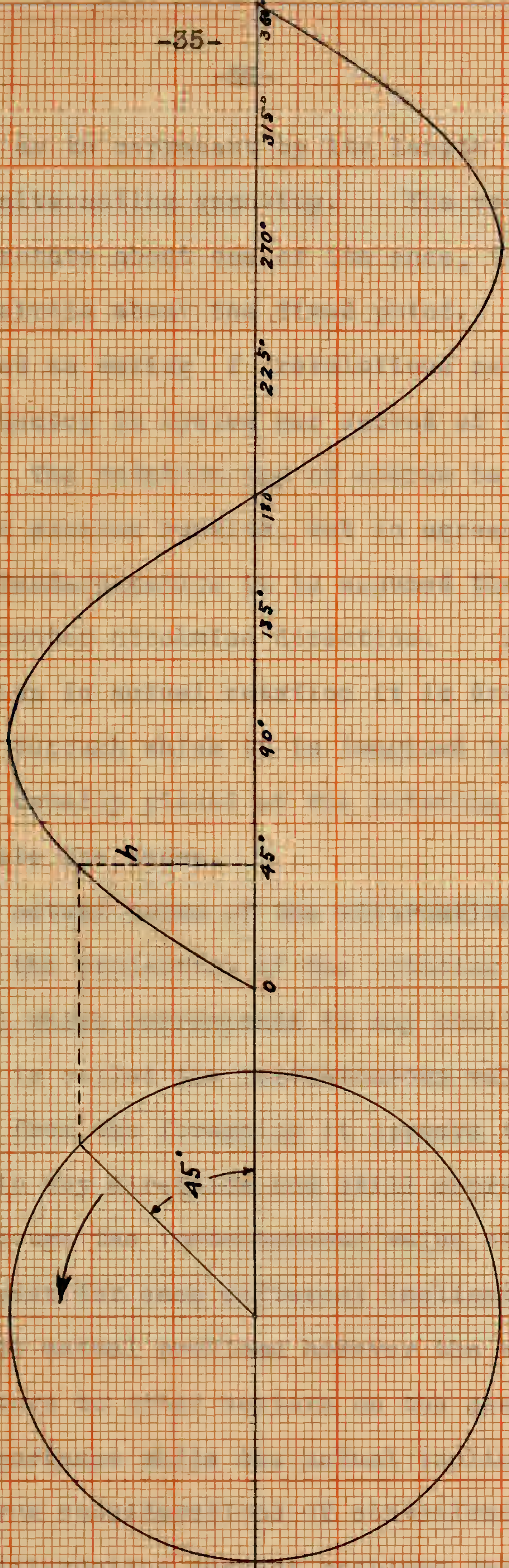


Fig. 8

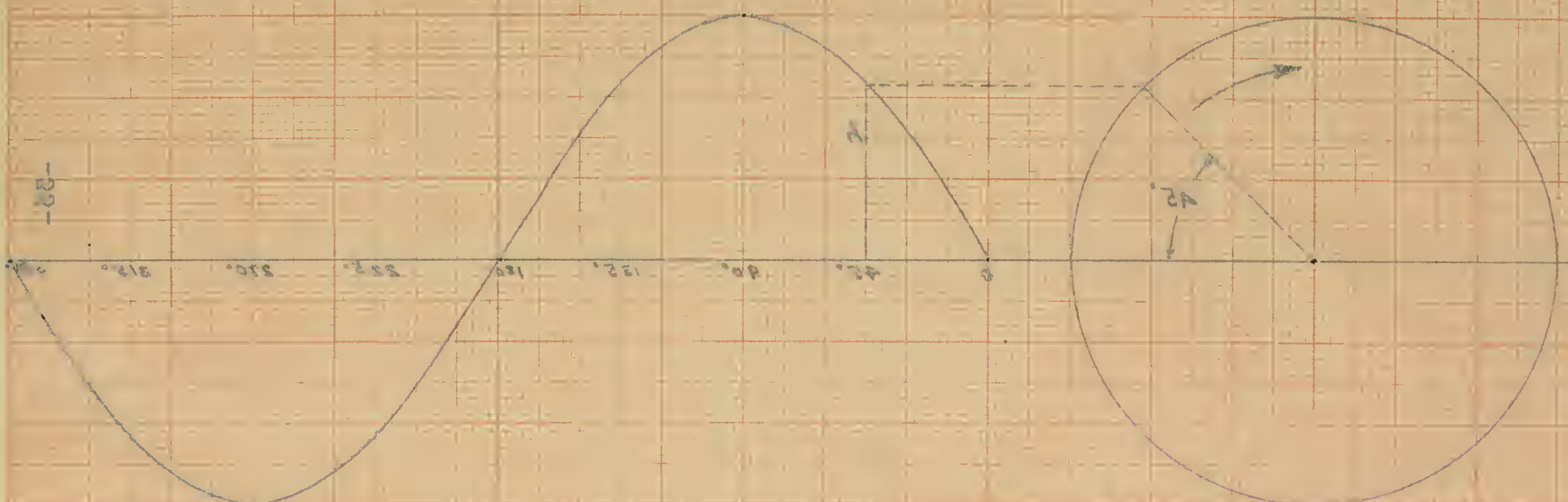


Fig. 8

some scale so as to represent by its length the maximum value of the alternating quantity. The vector is then conceived to rotate about one of its ends, the free end describing a circle about the fixed point. The moving end is imagined as making f revolutions per second, where f is the frequency in cycles per second of the quantity represented. The rotation may of course be taken in either direction with similar results, but in agreement with the A. I. E. E. standardization it is assumed that all vectors rotate in a counter clockwise direction. As the vector can not be drawn in actual rotation it is drawn in one of the positions through which it is imagined to pass. An arrow head is usually placed at the rotating end of the vector to designate the sense.

The actual value of the alternating quantity (determined from the projection of the rotating line as previously shown) which corresponds to any position of the rotating vector is called the instantaneous value of the quantity. Then from the foregoing it appears that a vector may be drawn in any direction and still represent the same quantity, although the instantaneous value of the quantity will be different for each different inclination of the vector. In most actual problems however the relative inclination of vectors to other vectors on the same diagram is of the utmost importance while the actual inclination is generally chosen from considerations of convenience.

In alternating current problems vectors are used in the form of vector diagrams to show the relation existing

between a number of alternating quantities. Thus for example it may be desired to show the effective or R. M. S. (root mean square) value and relative phase of the current resulting from having impressed upon a certain circuit an e. m. f. of a given effective value.

In general it is the effective values of current and voltage, rather than instantaneous and maximum values that are of interest. Further it is the phase difference between the various quantities and not their actual phase at any particular time which is of importance.

The three attributes of the vector quantities shown in the vector diagram, which it is of especial importance to represent are:

1. Virtual, or effective or R. M. S. value.
2. Relative phase.
3. Sense.

These are exactly the properties of the alternating quantity which may be obtained from the length, the inclination and the position of the arrow head of the rotating vectors when they are drawn to represent the quantities at any definite instant of time.

Thus for example, in figure 9, OA and OB represent the current and voltage in a circuit. The length of the lines gives the number of amperes and volts respectively, the angle θ gives the phase difference between the current and voltage and the arrow heads indicate their relative sense.

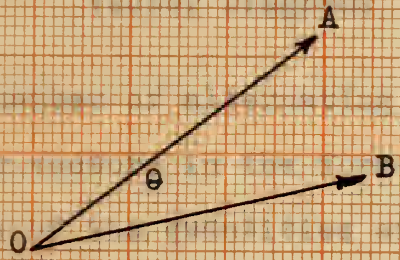


Fig. 9

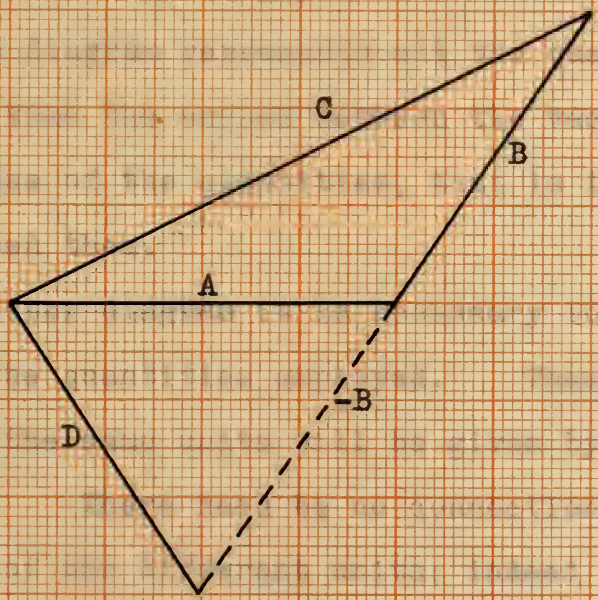


Fig. 10

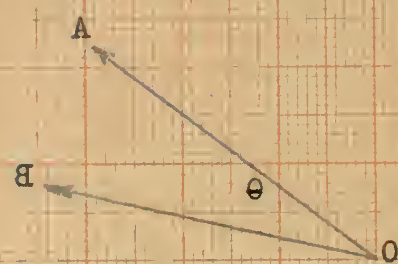


Fig. 9

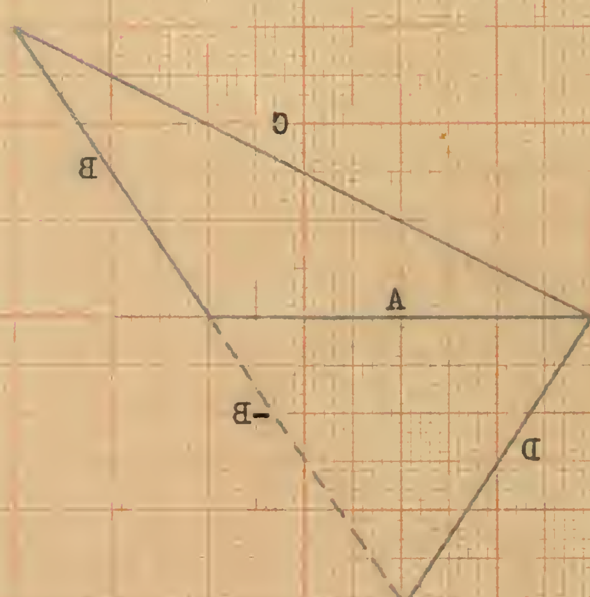


Fig. 10

Vector Diagrams

When a number of alternating quantities are represented by rotating vectors in the form of a vector diagram the maximum values of the quantities are very frequently used. In the case of simple harmonic quantities the effective value is a definite fraction of the maximum, that is, .707 times the maximum. The lengths of the lines on the vector diagrams are then proportional to the effective values of the quantities and may be taken to represent the effective values to a scale having the ratio of .707 to 1 to the original scale. In ordinary calculations it is best to consider that the lengths of the vectors represent the effective values and to choose the scale accordingly.

Since the diagram represents all the quantities at a given instant of time the angles between the vectors will be the relative phase of the quantities, that is the difference in phase between them.

In the vector diagram it is necessary to have a scale for each of the quantities employed. Then all quantities measured in the same units will be given by the vector to the same scale. There need be no connection whatever between the scales of the different units; indeed it is often convenient for the same vector to represent more than one quantity to their corresponding scales. For example a vector may represent a current to a scale of amperes and a voltage, that is current times resistance, to a scale of volts.

The scales in this case are chosen such as to make this possible, and will have a definite relation to one another determined by the resistance of the circuit.

Combination of Vectors

The addition and subtraction of vectors is treated in exactly the same way as the combination of forces in mechanics and needs very little discussion here. Figure 10 shows two vectors A and B. The addition of these vectors gives the vector C and their subtraction gives the vector D. Consider two rotating vectors a and b, figure 11, which are at an angle with each other. The sum of these two vectors may be represented by a third vector c which does not coincide with, that is, is not in phase with either a or b. If now there vectors are projected upon the moving line as previously pointed out they will generate three sine curves which are out of phase with each other. Upon investigation it is found that the curve traced by the vector c is the sum of the curves traced by the vectors a and b and that it has the same phase relation to curves a and b that the vector c has to the vectors a and b.

From figure 11 it is seen that the maximum values of the sine curves occur when the generating vector is in the vertical position and since the ordinate of the curve is the same as the length of the vector this length must necessarily be that of the maximum value of the varying function.

A problem in this connection will serve to impress

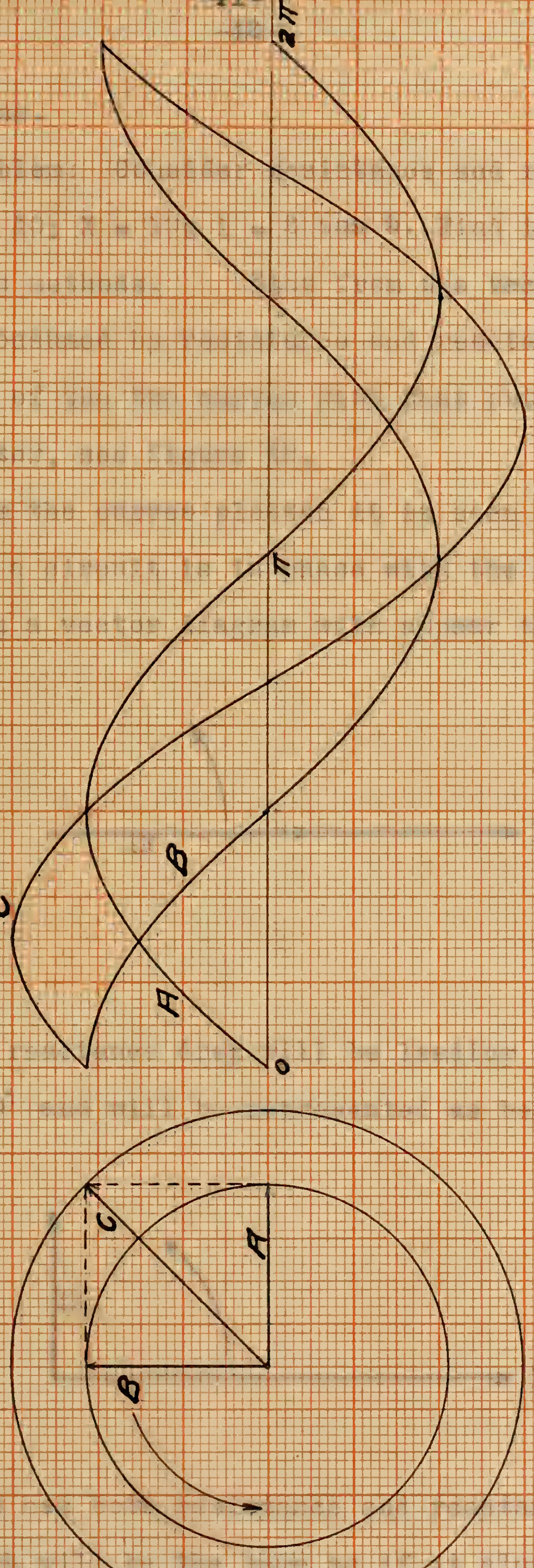


Fig 11
Waves Generated by Projection
of
Rotating Vectors A, B, C.

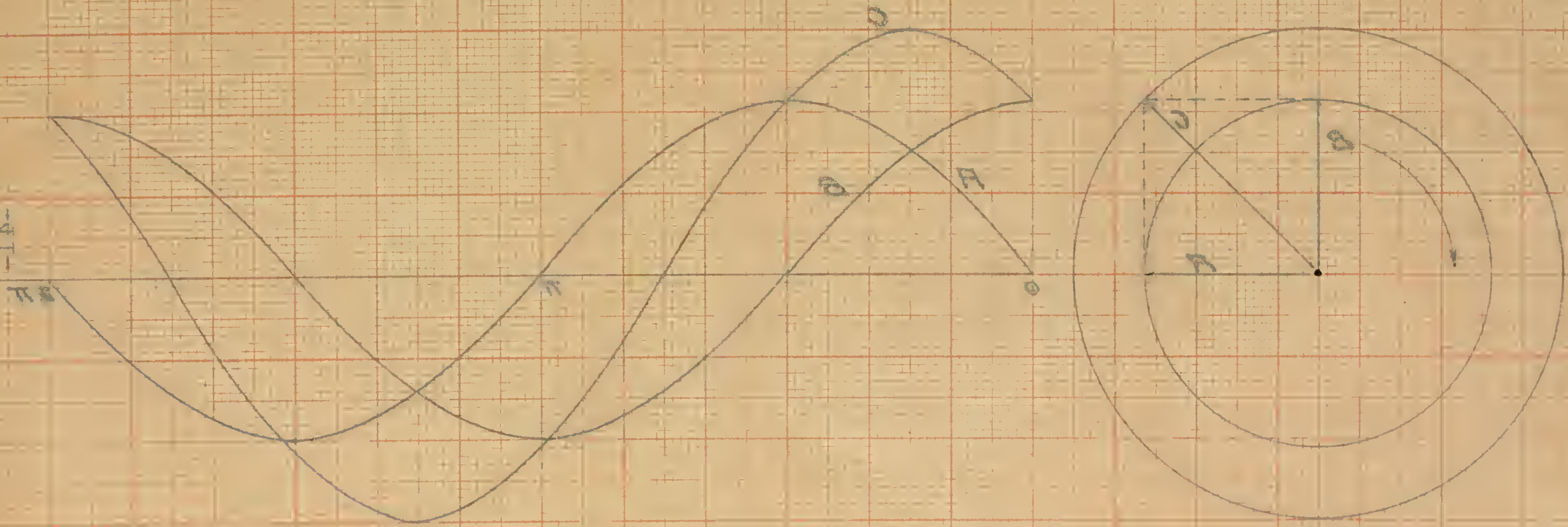


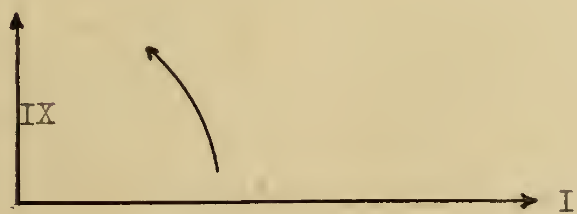
Fig 11
Waves Generated by Projection
of
Rotating Vectors A, B, C.

these relations.

Problem: Consider resistance and reactance in series. R = 10; X = 10; i = 2 cos θ. Find the impressed voltage by two methods. Plot from the vectors the waves of e. m. f. consumed by resistance and reactance and then check the sum of the two curves with that obtained from the resultant vector, see figure 12.

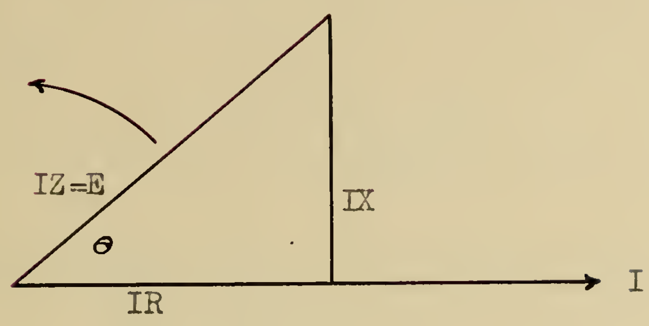
From the curves plotted it is seen that the resistance drop in a circuit is in phase with the current and if represented in a vector diagram will appear thus

Similarly the reactance drop will be leading the current by an angle of 90° and will be represented as below.

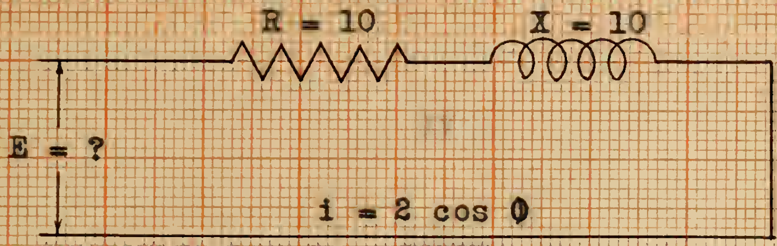


If the circuit has both resistance and reactance the separate effects of each will be the same as if it alone were in the circuit but acting together there will be a resultant effect.

Thus



In the above figures the vectors are considered as rotating in a counter clockwise direction as shown by the arrows according to which the current is lagging behind the impressed e. m. f. by some angle θ depending upon the relative values of r and x. A comparison with the sine curves previously plotted in figure 7, shows the same relations.



First Method.

$$I = \frac{E}{Z} = \frac{E}{\sqrt{R^2 + X^2}}$$

$$E = I\,Z = 2\sqrt{200}$$

$$= 28.3 \text{ volts}$$

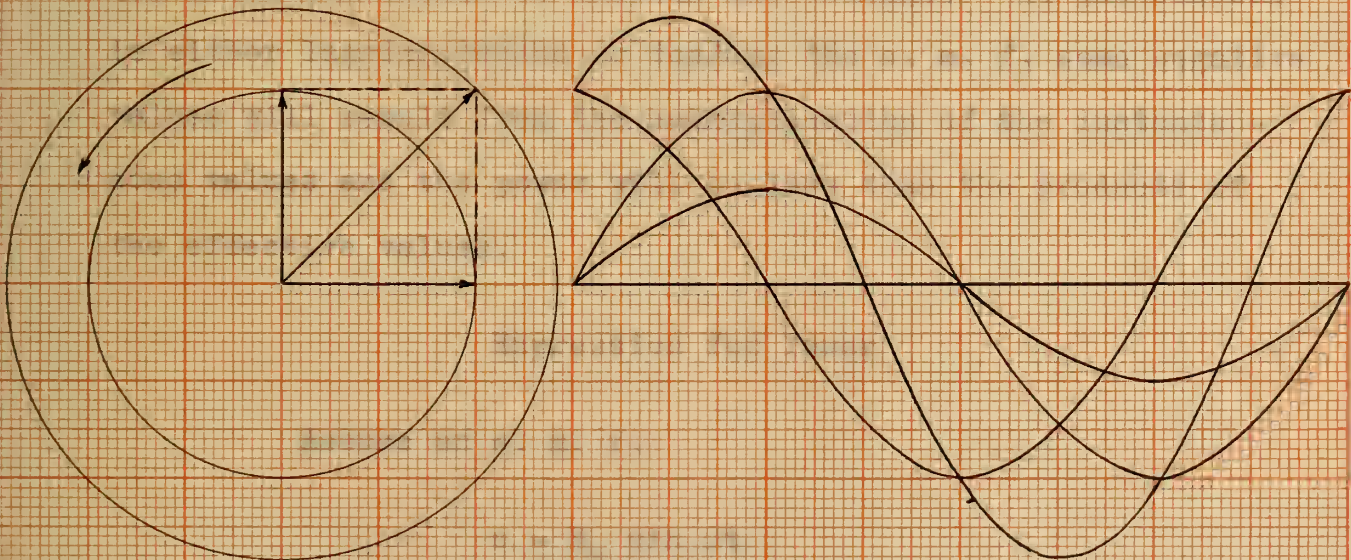
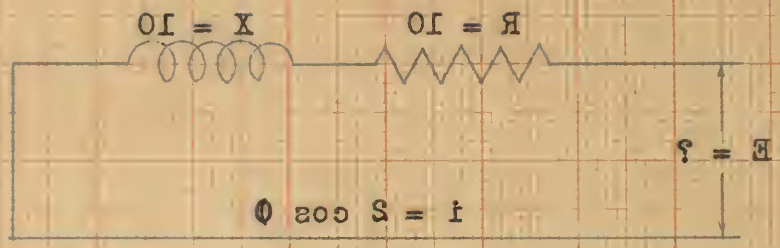


Fig. 12



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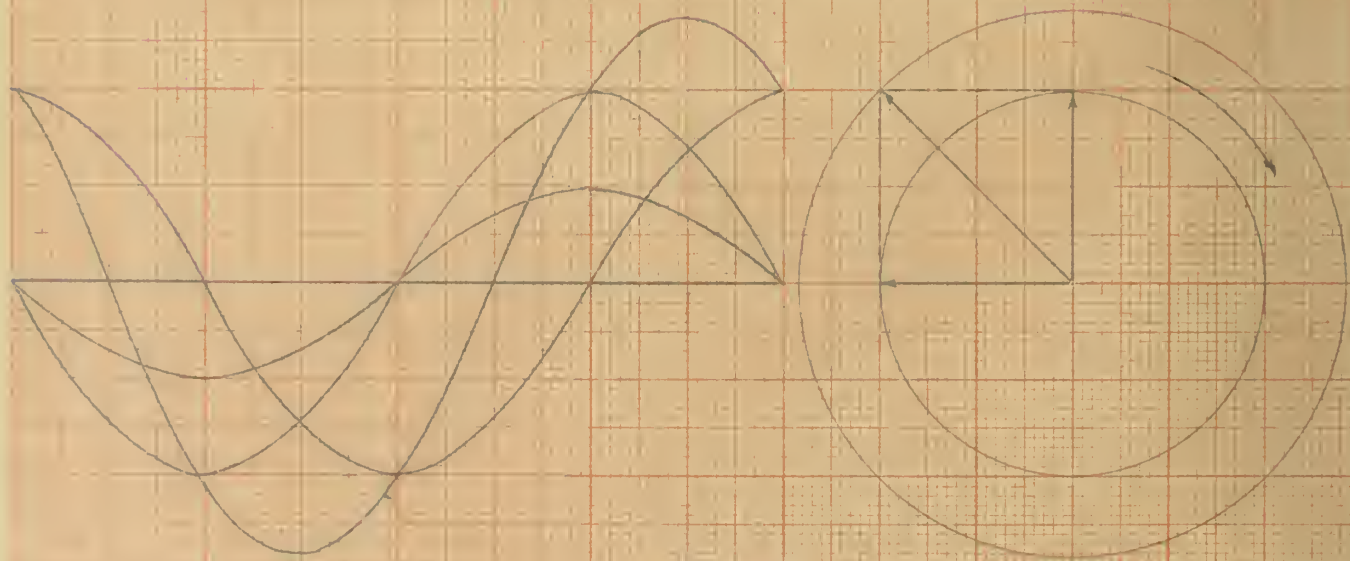

Fig. 12

IV

POWER

In direct current circuits the power is equal to the product of the e. m. f. and current but in alternating current work this is not in general the case. The power delivered at any instant is the product of the instantaneous values of e. m. f. and current and the average power is the average over a complete cycle of these instantaneous values. In case the current and e. m. f. are in phase the products of the instantaneous values will always be positive in which case the average power is a maximum and is the product of the effective values of current and voltage. If the current is either lagging behind or leading the e. m. f. some negative values will result from the multiplication of the instantaneous values and the power will be less than the products of the effective values.

Expression for Power

Assume an e. m. f.

$$e = E_m \sin \omega t$$

sets up a current

$$i = I_m \sin (\omega t - \theta)$$

The power at any instant is

$$e\ i = E_m \sin \omega t\ I_m \sin (\omega t - \theta)$$

$$= E_m\, I_m \cos\theta \quad (\sin^2 \omega t - \sin \omega t \cos \omega t)$$

and the average power for a complete cycle is the integral of the instantaneous powers divided by 2π or

$$\text{Average Power} = \frac{E_m\, I_m \cos\theta}{2}$$

$$= E\, I \cos\theta$$

Figure 13 represents sine waves of current and e. m. f. which are in phase and of the same maximum value. The product of the instantaneous values of e. m. f. and I takes the form of a sine wave which pulsates at twice the frequency of the original waves and has a maximum value of one half that of either the e. m. f. or current. In case the current and e. m. f. have maximum values of unity as shown in figure 13 the average power is .5 as is found by application of the formula just given or as follows.

Tabulation for Average Power

From Figure 13

θ	e	i	ei	θ	e	i	ei
0	0	0	0	115	.96	.96	.92
15	.26	26	.067	120	.86	.86	.75
30	.50	50	.25	135	.70	.70	.50
45	.70	70	.50	150	.50	.50	.25
60	.86	86	.75	165	.26	.26	.067
75	.96	.96	.92	180	.00	.0	0
90	1.00	1.00	1.00				

of + values = 5.974

of - values = 0.0

$$\text{Average Power} = \frac{5.974}{12} = .5$$

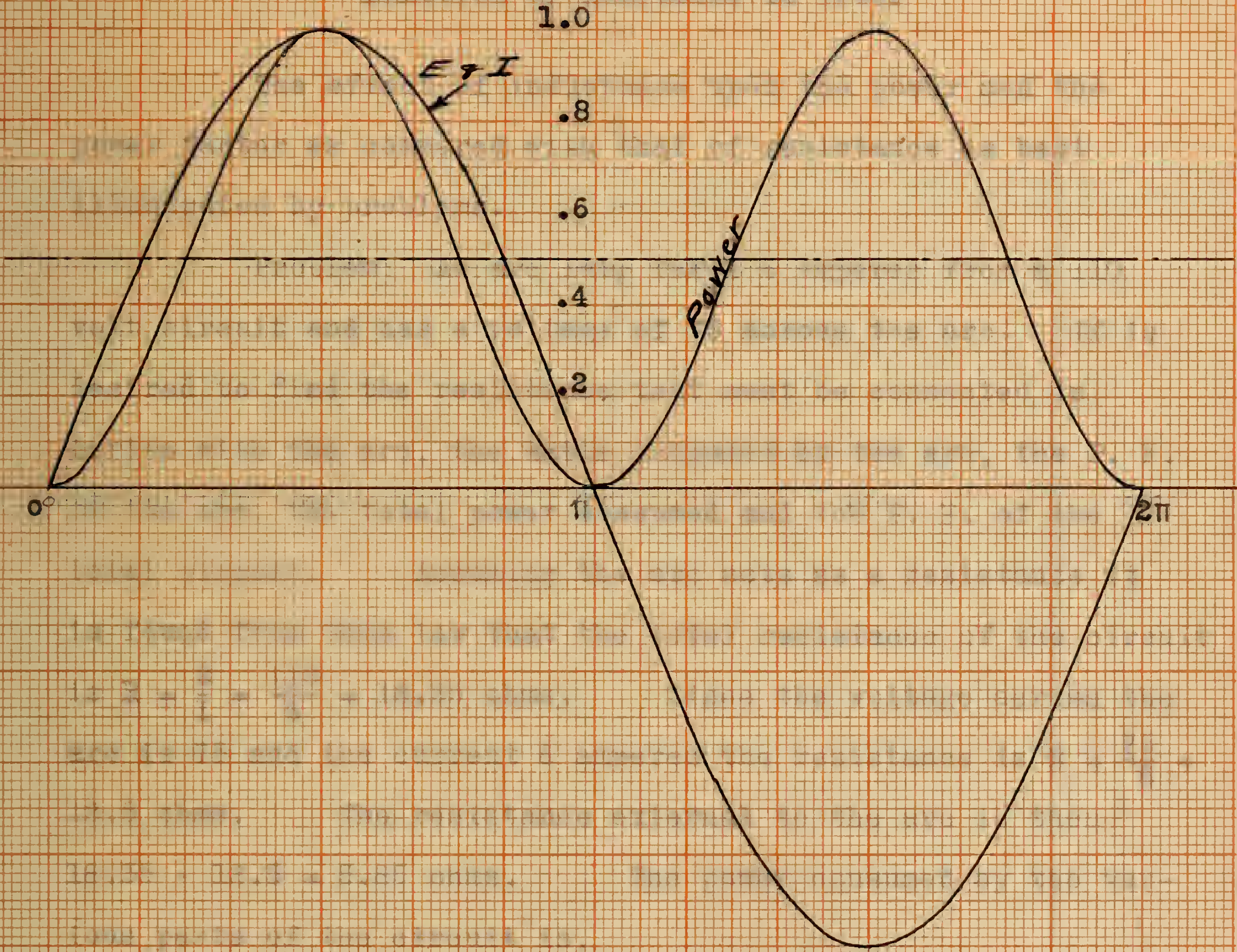

Fig. 13
E.M.F., current and power.
$E_m = 1 \quad I_m = I \quad P = .5$

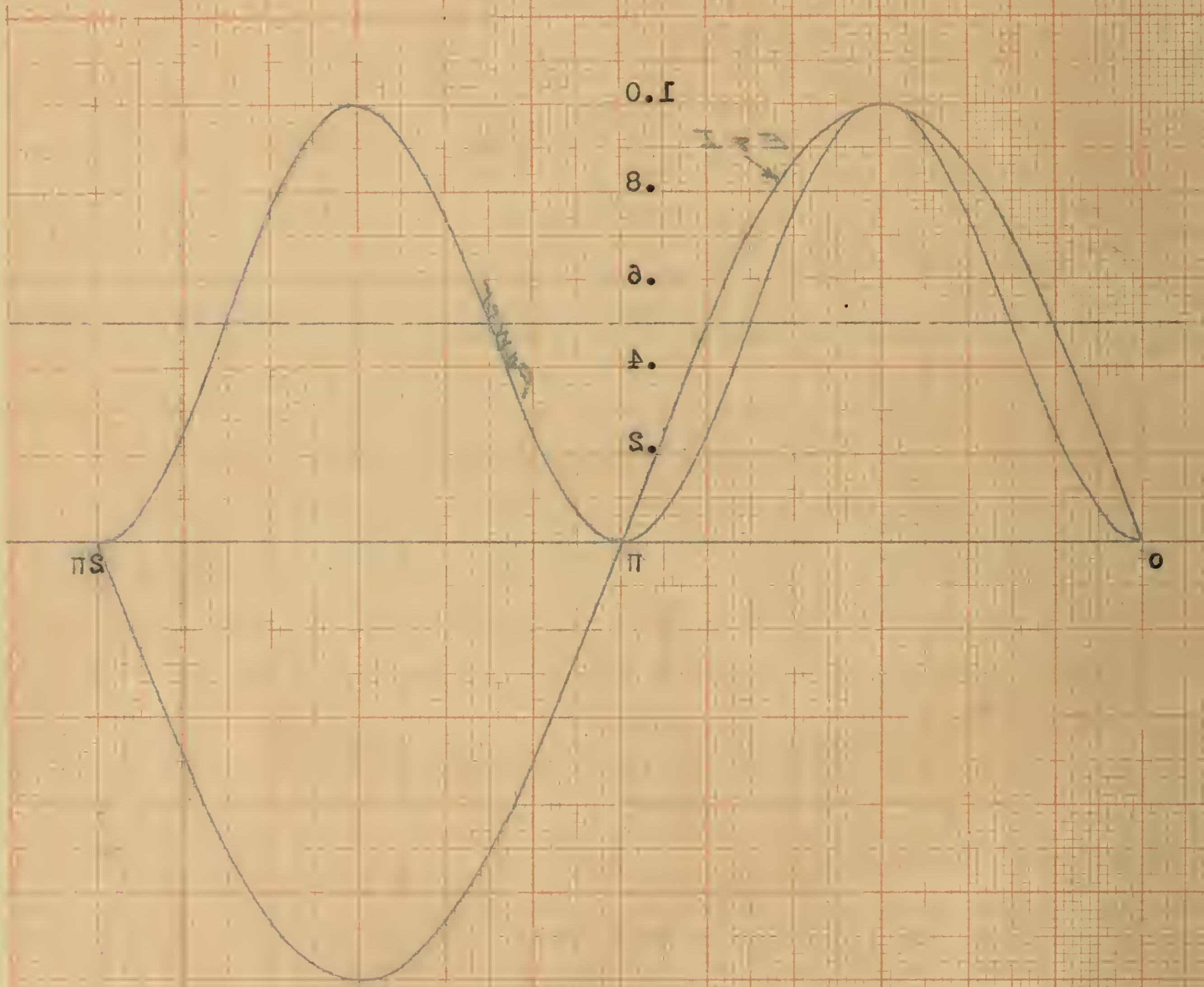

Fig. 13

E.M.F., current and power.

$E_m = 1 \quad I_m = 1 \quad P = .5$

Similarly if the current and voltage are at phase relation of 90 it is found that the negative and positive values of power are just equivalent and that the resultant average is zero.

Relation of Reactance to Power

The effect of inductance upon the power and the power factor as compared with that of resistance is best illustrated by problems.

Problem: An arc lamp takes 6 amperes from a 110 volt circuit and has a voltage of 75 across the arc. If is desired to find the resistance that must be connected in series with the arc, the watts consumed by the arc, the P. F. of the arc, the total power consumed and the P. F. of the total circuit. Assuming the arc acts as a resistance it is found from Ohms law that the total resistance of the circuit is $R = \frac{E}{I} = \frac{110}{6} = 18.35$ ohms. Since the voltage across the arc is 75 and the current 6 amperes the resistance is $R = \frac{75}{6} = 12.5$ ohms. The resistance external to the arc is then $18.35 - 12.5 = 5.85$ ohms. The power consumed by the various parts of the circuit is,

For arc,

$$\text{Power} = I^2 R = 6^2 \times 12.5 = 450 \text{ watts}$$

For resistance,

$$\text{Power} = I^2 R = 6^2 \times 5.85 = 210 \text{ watts}$$

and the total power consumed is 660 watts which is the product of the current and e. m. f. since the circuit contains only resistance.

The power factor, that is the relation of the true watts to the product of the current and e. m. f., is in this case unity as would be expected in a circuit of pure resistance. It is to be observed in this case that the power consumed by the auxiliary resistance forms a very appreciable part of the total watts consumed by the circuit. The vector diagram representing this circuit is given in figure 14, which shows that the total e. m. f. impressed is equal to the arithmetic sum of the e. m. f.'s consumed by the arc and the auxiliary resistance.

If now this resistance is replaced by a reactance of negligible resistance and of such ohmic value that the current remains constant at 6 amperes there will be some very marked changes in the properties of the circuit. If the current is to remain the same the total impedance Z must be the same as in the previous case, viz. Z = 18.35 ohms. Using the value of $R_{arc} = 12.5$ ohms it is found from $Z = \sqrt{R^2+X^2}$ that X = 13.4 ohms. The power consumed by the arc is the same as before, that is, 450 watts and the power consumed by the reactance is I^2R_X but since the value of R_X is zero the I^2R_X loss is zero.

The power factor in this case, the ratio of the true watts to the product of E and I, is

$$P.\ F. = \frac{450}{660} = .682$$

or

$$P.\ F. = \frac{R}{Z} = \frac{12.5}{18.35} = .682$$

That is, the P. F. is the ratio of the resistance to the total impedance. The vector diagram for this condition shows a

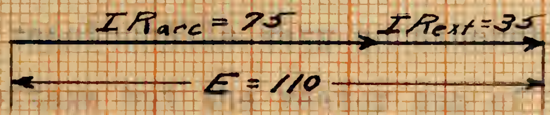

Fig. 14
Vector diagram for arc lamp when
voltage is reduced by resistance.

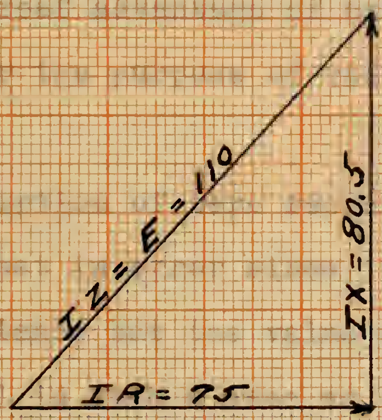

Fig. 15
Vector diagram for arc lamp when
voltage is reduced by reactance.

Fig. 14
Vector diagram for arc lamp when voltage is reduced by resistance.

Fig. 15
Vector diagram for arc lamp when voltage is reduced by reactance.

very marked difference from that of the preceding problem, see figure 15. The algebraic sum of the voltages across the arc and the reactance is very considerably greater than the impressed voltage but the vector sum of the I R and I X is just equivalent to the impressed e. m. f.

It is of especial interest to compare the values of total power consumed by the circuit in each instance. When the total e. m. f. is cut down to arc e. m. f. by resistance the power lost thereby is quite large, viz. 210 watts but when the same effect is obtained by the use of resistance less reactance the power lost is zero. This shows very conclusively the great advantage of using reactance rather than resistance for the purpose of reducing an excessive e. m. f.

The assumption of zero resistance in the reactance is of course somewhat in error since it is impossible to have a wire of no resistance but the value of this resistance is usually quite small in such a case as that cited and might be neglected without serious error.

This may be illustrated and some further instructive data may be obtained by calculating the reactance just assumed.

If the coil is considered as made of #10 wire wound about an iron core having an air gap of .5 inch and a cross sectional area of 1.5 square inches and if the reluctance of the iron is neglected the constants of the coil necessary to give the required reactance may be very readily calculated.

From the preceding problem x = 13.4 ohms at 60 cycles.

$$x = 2 \pi f L$$

$$L = \frac{13.4}{377} = .0355 \text{ Henrys}$$

$$L = \frac{.4 \pi N^2 \mu A 10^{-8}}{\ell}$$

$$N^2 = 371000$$

$$N = 608$$

The mean length of one turn is then 5 inches from which

$$R = \frac{5 \times 608}{12 \times 1000} = .254 \text{ ohms}$$

The resistance loss in this case is

$$\text{Watts} = I^2 R$$

$$= 6^2 : .254$$

$$= 9.15$$

from which it is seen that if negligible resistance had been assumed the error would not have been over 3 percent. The loss of 9 watts in this case as compared with the loss of 210 watts when the e. m. f. was reduced by resistance shows very conclusively the great advantage of using reactance rather than resistance for such purposes.

V

EFFECTS DUE TO RESISTANCE, INDUCTANCE AND CAPACITY

Combination of Resistance and Reactance

In figure 16 curves are plotted showing the variation of current with reactance in a circuit which has constant resistance and constant impressed e. m. f.

It is to be noted that in the circuit of 10 ohms resistance the addition of reactance has much less effect on the current than in the case of the 5 ohm circuit. This may be understood if it is remembered that the Ix drop is at right angles to the I R drop and at small values of Ix the vector sum of the two is only slightly greater than the resistance drop and hence the reactance has very little effect upon the current.

It is to be especially noted that if resistance is added the effect is very much greater. To illustrate, assume an e. m. f. of 20 volts impressed upon a circuit of 5 ohms resistance. The current in this case is $I = \frac{E}{R} = 4$ amp. If now a reactance of 1 ohm is added the total impedance is

$$Z = \sqrt{r^2 + x^2} = \sqrt{5^2 + 1^2} = 5.1$$

and the current flowing is

$$I = \frac{E}{Z} = \frac{20}{5.1} = 4.92 \text{ amp.}$$

If however a resistance of 1 ohm were added the total impedance would be 6 ohms and the current flowing $I = \frac{20}{6} = 3.34$ amp.

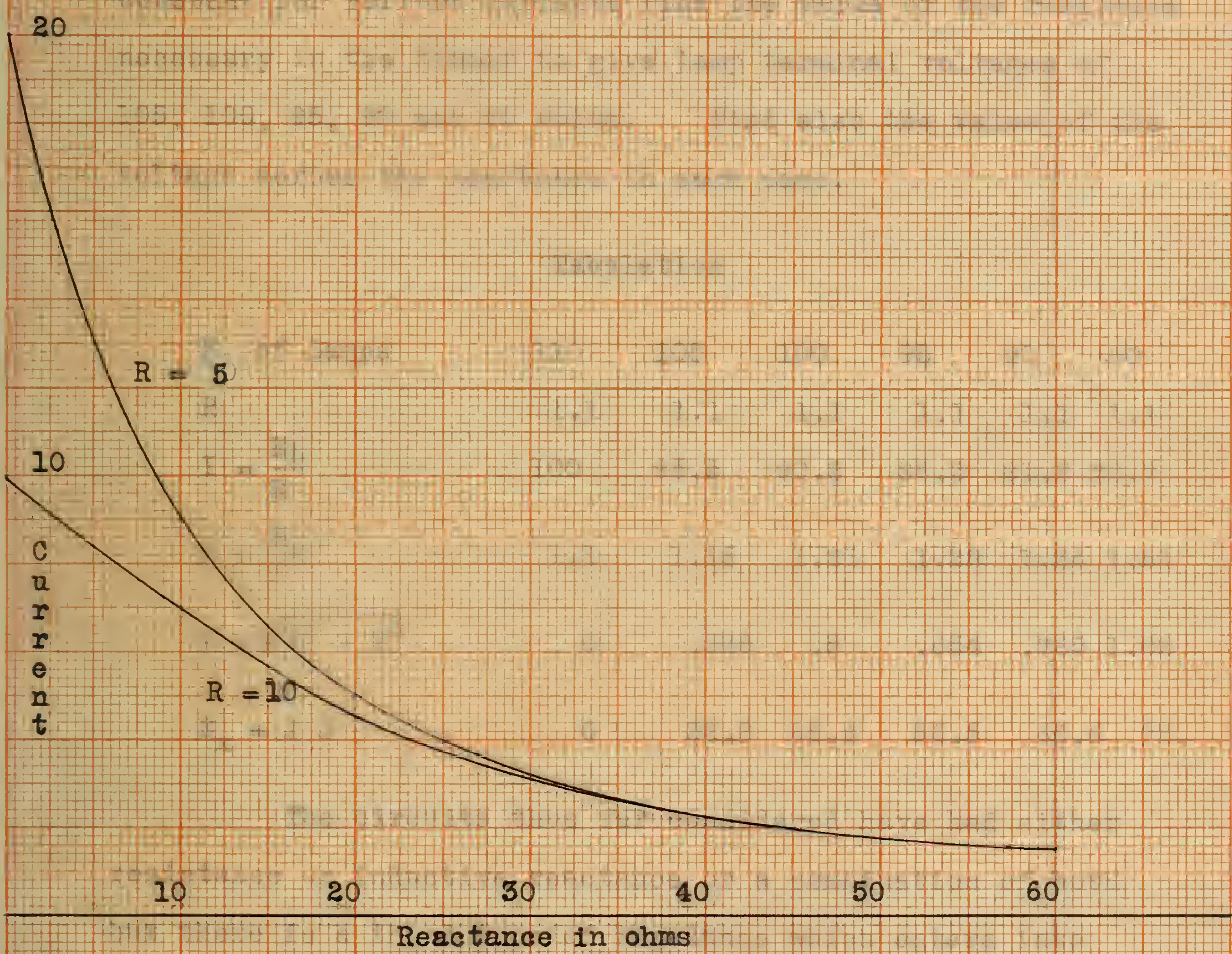

Fig. 16
Relation of current to reactance
in a circuit having constant
resistance and constant impressed
voltage of E = 100

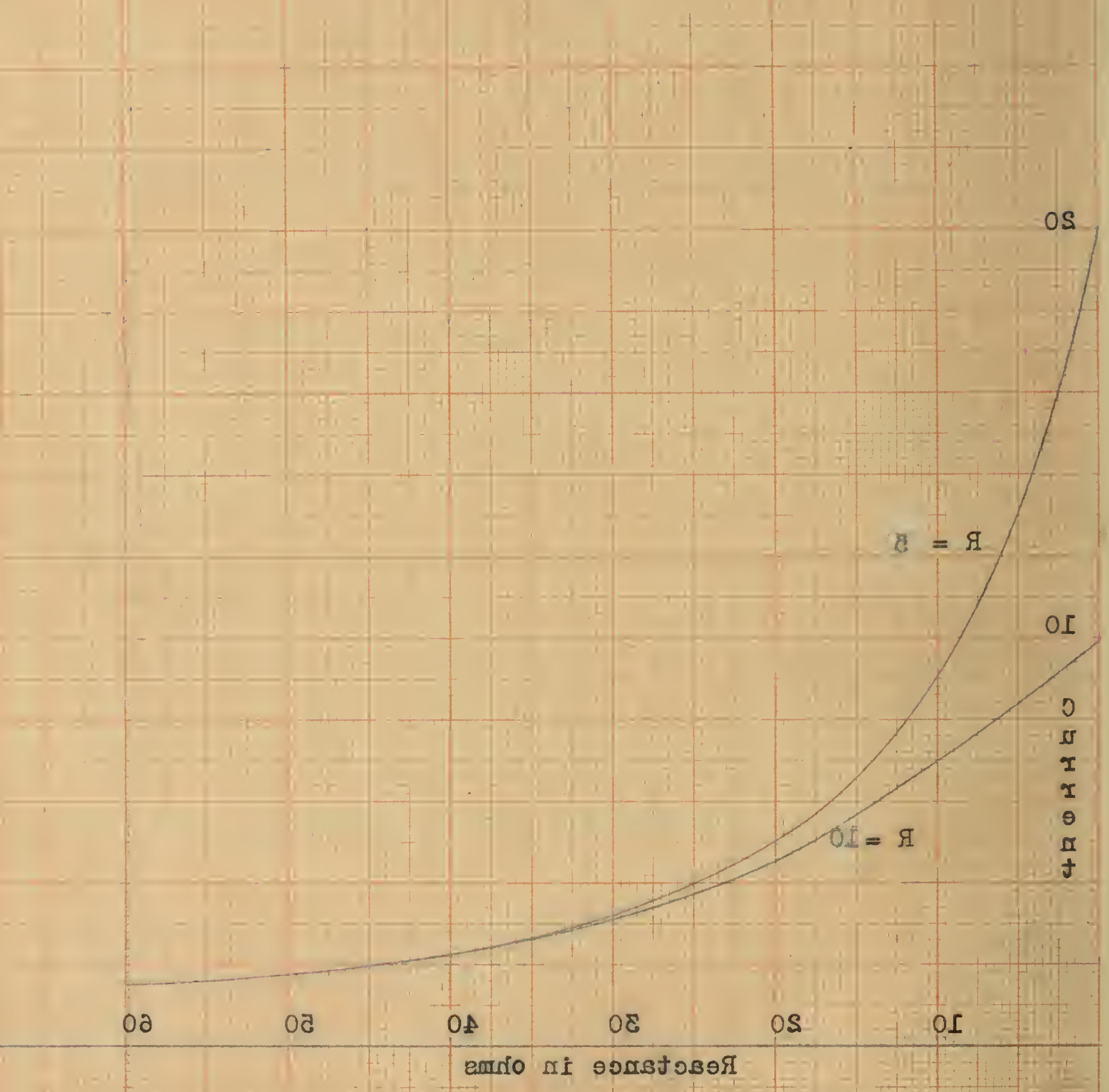

Fig. 16
Relation of current to reactance
in a circuit having constant
resistance and constant impressed
voltage of E = 100

Problem: A theater dimmer is to control the voltage on 200-16 candle power 110 volt incandescent lamps. Assuming that the resistance of each lamp is 220 ohms and that it remains constant for various currents find the value of the reactance necessary in the dimmer to give lamp terminal voltages of 105, 100, 95, 90 and 80 volts. Find also the value of the voltage across the reactance in each case.

Tabulation

E. of Lamps	110	105	100	95	90	80
R	1.1	1.1	1.1	1.1	1.1	1.1
$I = \frac{E_L}{R}$	100	95.5	90.8	86.3	81.8	72.7
$Z = \frac{E_t}{I}$	1.1	1.15	1.21	1.28	1.34	1.51
$X = \sqrt{Z^2 - R^2}$	0	.346	.5	.654	.765	1.03
$E_x = I X$	0	32.8	45.4	56.5	62.6	75.

The circuits thus far considered have had either resistance or inductive reactance or a combination of both but there is a third type of impedance which enters into alternating current calculations known as capacity reactance, that is the reactance due to a condenser.

Reactance Due to Capacity

The quantity of electricity stored in a condenser is dependent upon its capacity and the impressed voltage. That is, $Q = c\ e$ in which Q is the quantity of energy stored, c is the capacity in farads and e the e. m. f. impressed. This

stored energy tends to maintain an e. m. f. opposing the impressed e. m. f. hence the equation may be written

$$e_c = - Q\frac{1}{c}$$

In which e_c is the e. m. f. "set up" by the capacity. The quantity of energy stored at any instant is $q = \int idt$ which, when substituted in the equation for e_c gives

$$e_c = - \frac{1}{c} \int idt$$

or

$$d\, e_c = - \frac{1}{c}\ idt$$

If the current $i = I_m \cos \omega t$ is flowing in a circuit having capacity reactance but zero resistance it is evident that $-e_c = e_{imp.}$. Substituting for current in the expression for de_c above

$$d\, e_c = - \frac{1}{c}\ I_m \cos \omega t\ \ dt$$

from which

$$-e = e_c = - \frac{I_m}{c\ \omega} \sin \omega t$$

or

$$e = \frac{I_m}{c\ \omega} \cos\ (\omega t - 90)$$

If $\cos\ (\omega t - 90) = 1$, then

$$e = E_m$$

or

$$E_m = \frac{I_m}{c\ \omega} \quad \text{or } \sqrt{2}\ E = \frac{I\sqrt{2}}{c\ \omega}$$

or

$$E = \frac{I}{c\ \omega} \quad \text{or } \frac{E}{I} = \frac{1}{c\ \omega} \text{ ohms.}$$

That is the reactance due to capacity varies inversely as the

product of the capacity and the factor ω, which as previously shown is, $\omega = 2\pi f$.

If the current in a circuit containing a condenser changes as $i = I_m \cos \omega t$ the quantity of electricity being stored in the condenser will vary as $q = Q \cos \omega t$. The quantity at any time is

$$q = \int i dt$$

from which

$$\frac{dq}{dt} = i.$$

That is, the current is proportional to the derivative of the quantity and hence is $90°$ ahead of q in phase. From $q = c e$ it is evident that the e. m. f. consumed by the capacity $e = \frac{q}{c}$ is in phase with the varying q, or $90°$ behind the current.

Vector Diagrams

The vector diagram of a circuit containing capacity will then show a vector proportional to the e. m. f. of capacity and displaced $90°$ from the current. The vector diagram of a circuit containing resistance and capacity reactance is shown in figure 17 and the diagram for a circuit having resistance, inductance and capacity reactance is shown in figure 18. The power factor in each case is of course the relation of the true watts to the product of E and I, which is also the ratio of $\frac{IR}{E}$ as found for inductive circuits.

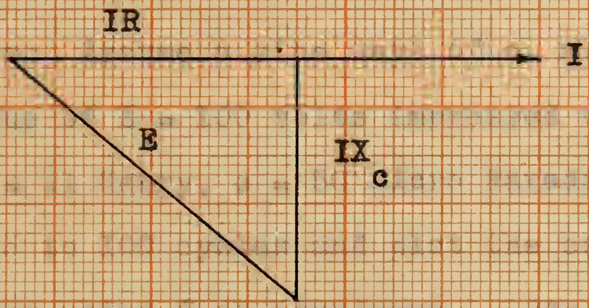

Figure 17.

Vector diagram for a series circuit having resistance and capacity.

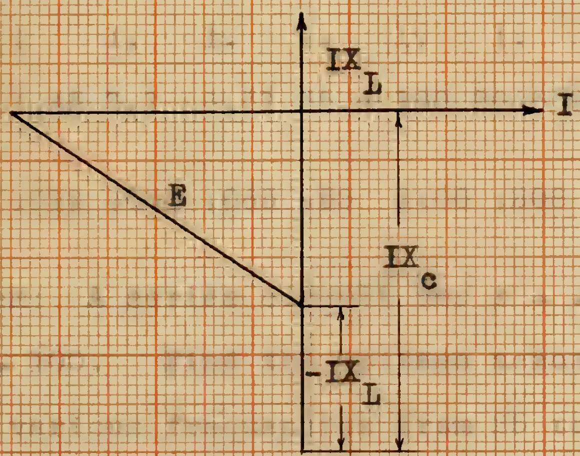

Figure 18.

Vector diagram for a series circuit having resistance, inductance and capacity.

Figure 17.

Vector diagram for a series
circuit having resistance and
capacity.

Figure 18.

Vector diagram for a series
circuit having resistance,
inductance and capacity.

Resonance of Series Circuits

Problem: Assume a sine wave of e. m. f. having an effective value of E = 100 volts impressed upon a circuit having r = 1, L = .1 Henry, c = 50 Micro Farads. Vary the frequency from 25 to 100 cycles and plot the curve of current and frequency using f as abscissae. See figure 19.

Tabulation

E	100	100	100	100	100	100	100	100	100	100
f	25	35	50	60	70	71.1	75	80	100	115
x_L	15.7	22	31.4	37.7	44.	44.7	47.1	50.2	62.8	72.1
x_c	127.3	90.0	63.6	53.	45.5	44.7	42.4	39.8	31.8	27.6
r	1.	1.	1.	1.	1.	1.	1.	1.	1.	1.
$I = \frac{E}{\sqrt{r^2+(x_L-x_c)^2}}$	.897	1.45	3.1	6.51	55.4	100	20.8	9.52	3.22	2.25
P.F.	.009	.014	.031	.065	.55	1.00	.208	.095	.032	.022

Problem: A series circuit has r = .2, L = .3 henry, c = 30 M.F., E = 100. Find the voltage across each part of the circuit for various frequencies from 25 to 100. Plot curves of voltage and frequency for each part of the circuit.

Tabulation

E	100	100	100	100	100	100	100	100	100	100
f	25.	40	50	53	55	60	70	80	90	100
x_L	47.	75.4	94.3	100	103.8	113.2	132.	151.	169.8	188.5
x_c	212.	132.6	106.	100	96.4	88.5	75.7	66.2	59.	53

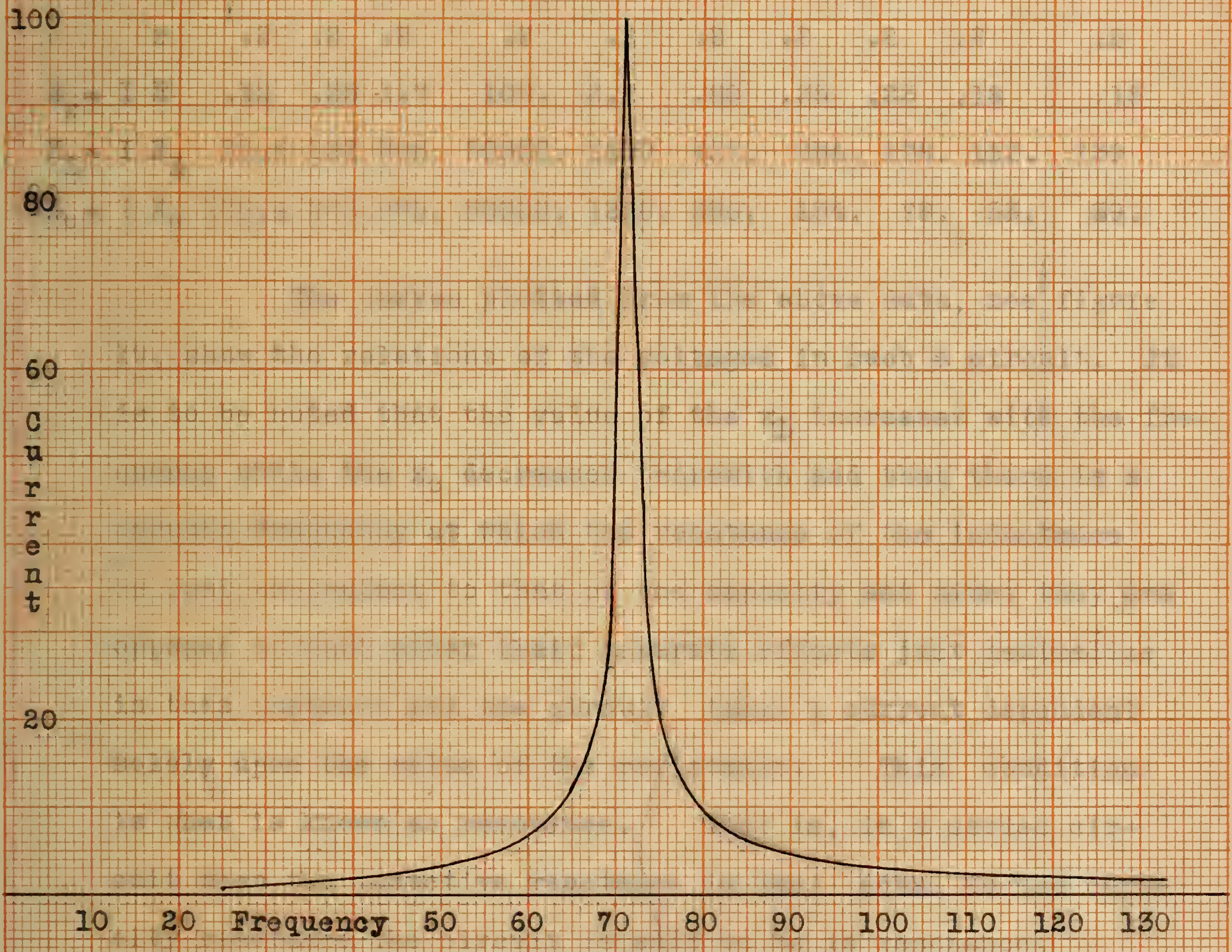

Fig. 19

Relation between frequency and
current in a series circuit having
resistance, inductance and capacity.

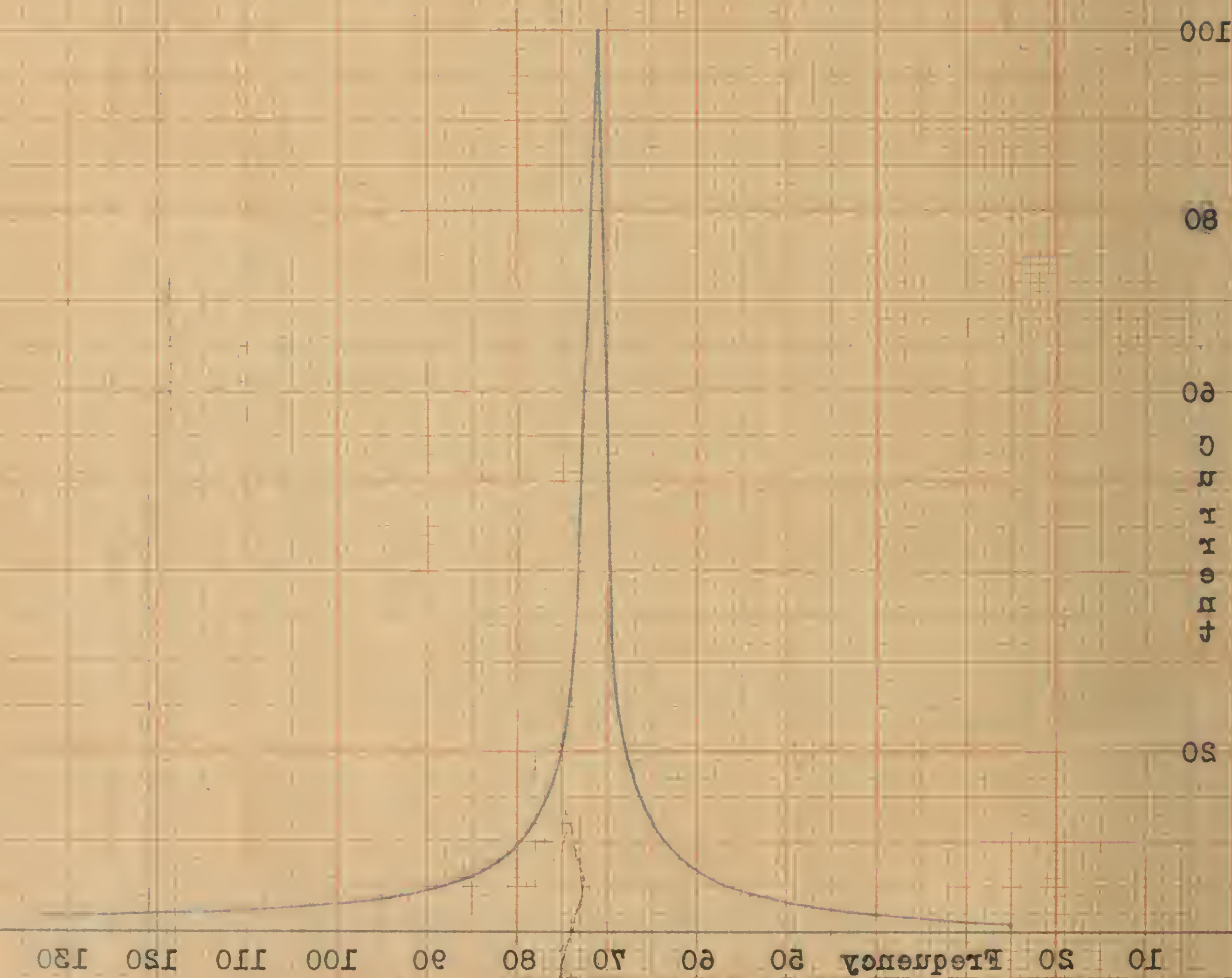

Fig. 19

Relation between frequency and
current in a series circuit having
resistance, inductance and capacity.

f	25.	40	50	53	55	60	70	80	90	100
$x_L - x_c$	-165	-57.2	-11.7	0	7.4	24.7	56.3	84.8	110.8	135.5
$I = \frac{E}{\sqrt{r^2 + (x_L - x_c)^2}}$	.60	1.75	8.53	500.	13.5	4.04	1.77	1.18	.90	.74
r	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2
$E_r = I\,R$	.12	.35	1.7	100.	2.7	.80	.35	.23	.18	.15
$E_L = I\,X_L$	28.5	132	805	50000.	1400	457.	234.	178.	153.	139
$E_c = I\,X_c$	128.4	233.	905.	50000.	1288.	356.	134.	78.	53.	39.

The curves plotted from the above data, see figure 19, show the relations of the voltages in such a circuit. It is to be noted that the value of the x_L increases with the frequency while the x_c decreases therewith and that there is a certain frequency at which the reactance of the inductance is just equivalent to that of the capacity and since they are opposed to each other their separate effects just neutralize in this instance and the circuit takes a current dependent solely upon the value of the resistance. This condition is what is known as resonance. That is, in a series circuit when the inductive reactance is just equal to the capacity reactance the circuit is said to be in resonance.

It is of further importance to note the variation of the voltages across the reactances at the different frequencies. Thus in the circuit represented in figure 20 the voltages across the inductance and capacity are each 50000 volts while the impressed voltage on the circuit as a whole is only 100. This phenomenom of the increase of voltage due to resonance is of great importance in electric circuits,

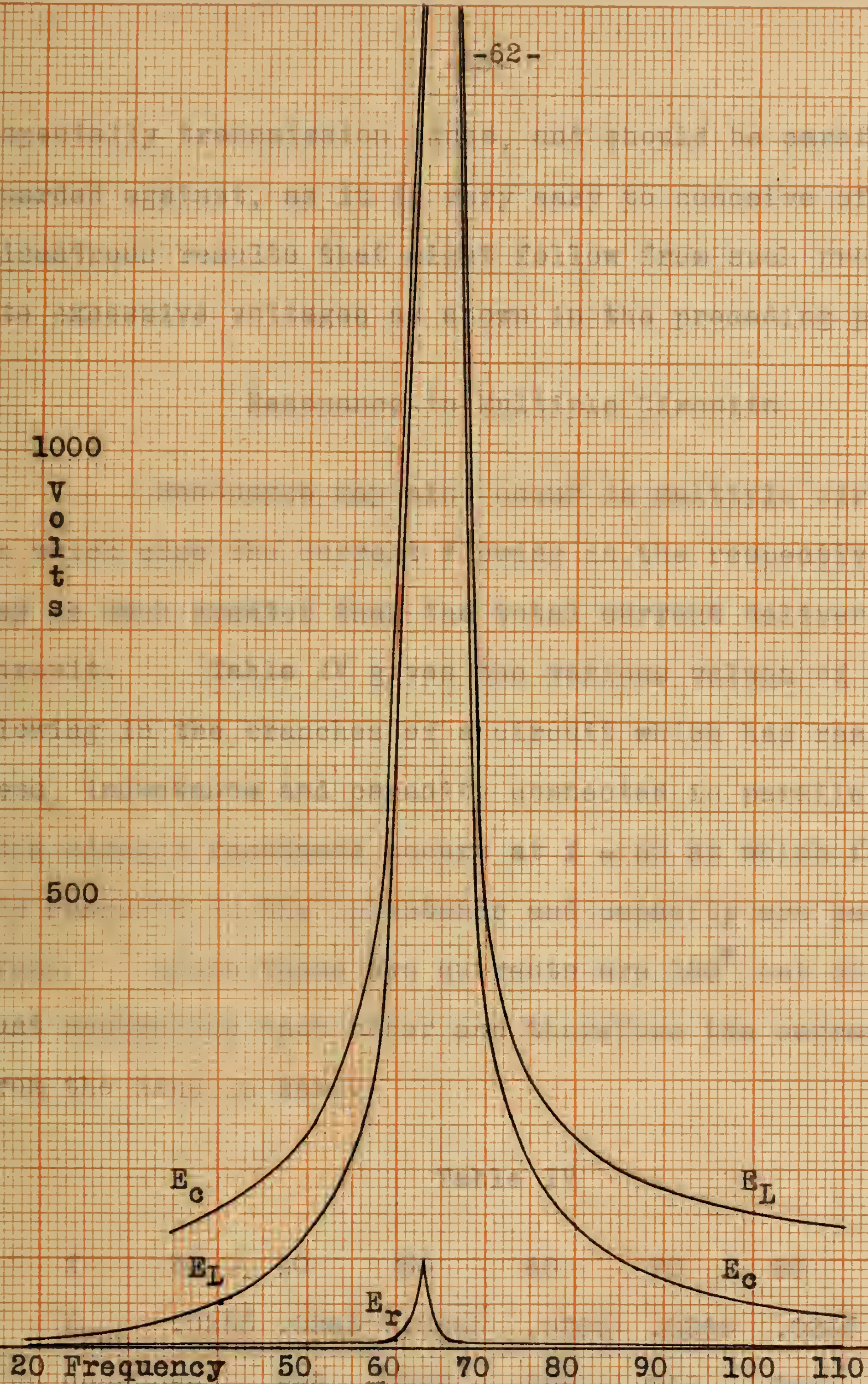

Fig. 20

Voltage across sections of a circuit containing resistance, inductance and capacity.

Fig. 20

Voltage across sections of a circuit containing resistance, inductance and capacity.

especially transmission lines, and should be carefully guarded against, as it is very easy to conceive of the disastrous results that might follow from such proportionate excessive voltages as shown in the preceding problem.

Resonance in Multiple Circuits

Resonance may also occur in multiple circuits, in which case the current flowing in the respective branches may be much greater than the total current delivered to the circuit. Table IV gives the various values of current flowing in the branches of a circuit which has resistance less inductance and capacity connected in parallel. In this circuit resonance occurs at f = 60 at which frequency the currents in the inductance and capacity are each 10 amperes. Since these two currents are 180° out of phase they just neutralize each other and therefore the current taken from the line is zero.

Table IV

f	30	40	50	60	70	80	90
$L_{Hen.}$	.0265	.0265	.0265	.0265	.0265	.0265	.0265
C_{MF}	265.	265	265	265.	265.	265.	265.
x_L	5.	6.66	8.32	10.	11.65	13.3	14.9
I_L	-20.	-15.	-12.	-10.	-8.55	-7.5	-6.7
x_c	20.	15.	12.	10.	8.55	7.5	6.7
I_c	5.	6.66	8.3	10.	11.65	13.3	14.9
I_T	-15.	-8.34	-3.9	0	3.1	5.8	8.2

In reality the above condition could not quite occur since there must necessarily be some resistance included in the inductive branch, in which case there must be a component of current in phase with the impressed e. m. f.

In general the condition for resonance of parallel circuits is that the susceptance of the inductive branch is equivalent to the susceptance of the capacity.

Or

$$b_L = b_c$$

or

$$\frac{X_L}{r^2_L + x^2_L} = \frac{X_c}{r^2_c + x^2_c}$$

See Pages 84 and 85.

Energy Stored in Inductance

Inductances and condensers are so widely used that it is of interest to know the energy that may be stored in them under certain conditions of operation.

Consider for example a circuit containing resistance and inductance and having impressed an e. m. f. E. The e. m. f. impressed must be entirely consumed in the circuit and hence the expression may be written

$$e = i\, r + L \frac{di}{dt}$$

Multiplying by i

$$ei = i^2\, r + L\, i \frac{di}{dt}$$

which when integrated between t and t_1 gives, if, when t = o, i = o and when t = T, i = I

$$\int_0^T e\, i\, dt = \int_0^T i^2\, r\, dt + \int_0^I L\, i\, di$$

$$\int_0^T ei\,dt = i^2\,r\,T + \frac{LI^2}{2}$$

and since $\int_0^T ei\,dt$ is the total energy dissipated from 0 to T, and $i^2\,r\,T$ is the energy consumed by the resistance in this time the difference of these two is the energy not yet dissipated or it is that stored in the inductance which as seen from the equation is

$$\text{Energy Stored} = \frac{1}{2}\,L\,I^2 \text{ jouls.}$$

Energy Stored in Condensers

In the case of a condenser having capacity c and impressed voltage e the quantity of energy stored is

$$q = c\,e.$$

The quantity being stored at any time t is idt and the quantity stored at any time is $\int idt$.

From which

$$\int idt = c\,e \quad \text{Differentiating and multiplying}$$

by e

$$e\,i\,d\,t = e\,c\,d\,e.$$

Then if

$$t = 0 \qquad e = 0 \quad \text{and if}$$

$$t = T \qquad e = E$$

$$\int_0^T c\,i\,dt = \int_0^E c\,e\,d\,e$$

$$= \frac{c\,e^2}{2}\Big]_0^E = \frac{1}{2}\,C\,E^2$$

which is the energy stored at any time T after the circuit has been closed.

Voltage Induced by Stored Energy

The energy stored in a magnetic field may, under certain conditions, set up a very high voltage across some part of the circuit. For example consider a coil of wire of which $L = .05$ Henry, connected in series with a resistance of r ohms. Assume that there is impressed a direct current voltage of such value that the current flowing is 15 amperes. Also assume a capacity of 1. microfarad connected in shunt with the inductance. If the circuit is opened instantaneously the energy stored in the inductance just equals that which will be stored in the capacity an instant after the switch is opened. Then from the previous equations

$$\frac{1}{2} L I^2 = \frac{1}{2} C E^2$$

which gives

$$E = 3450 \text{ volts.}$$

Similarly if the current flowing is 100 amperes the voltage across the capacity would be 22,400.

VI

COMPLEX QUANTITIES

Addition and Subtraction

In the application of mathematics it is found that the process of addition can always be carried out but the process of subtraction is sometimes possible and sometimes impossible. For example two men plus four men is six men and similarly any given number of men plus any other given number of men gives a certain definite number which will always have a meaning. Two men plus five men gives a definite number but two men minus five men gives minus three men, that is, the result is impossible, since there is no such thing as a negative man.

If distances are considered it is found that in general the processes of addition and subtraction may be carried out but that there are some cases in which the latter is impossible. For example if one starts to walk to the right from a given point and walks 10 feet and then pauses and again walks 15 feet in the same direction the total distance from the starting point is 25 feet, but if on the other hand after reaching the 10 foot point the person should turn about and walk 15 feet in the opposite direction he would be, if there is no insurmountable obstacle in the way, 5 feet from the starting point, but on the opposite side of it. If then

the distances from the object to the right are considered as positive and the distances to the left as negative the person is then negative 5 feet from the object or as usually expressed, his distance from the object is -5 feet. In this case the negative number is just as real as the positive number and has just as much meaning, but in the case of the men previously cited it has no meaning whatever. This shows that the negative number is a mathematical and not a physical conception and that physically it may or may not be represented as the case may be.

Multiplication

In the operation of multiplication it is found that if two factors having positive signs are multiplied together the resulting number is positive but it is also found that if two factors each having negative signs are multiplied together the resulting number has a positive sign. If however one positive number is multiplied by a negative number the result is a negative number, that is -1 times 2 is -2 which if applied to the location of a point would mean that the point in question is -1 times + 2 units or -2 units to the left of the origin or reference point.

If the factor -1 had not been introduced it would have been understood that the point in question was 2 units to the right of the origin. Multiplication by -1 then reverses the direction, or rotates the point through 180° .

Multiplication by $\sqrt{-1}$, Quadrature Numbers

If the number +2 is multiplied by $\sqrt{-1}$ the result is $+2\sqrt{-1}$ which has seemingly no meaning since the $\sqrt{-1}$ is usually spoken as an imaginary number. A second multiplication by $\sqrt{-1}$ gives +2 times $\sqrt{-1}$ times $\sqrt{-1}$ or is equal to +2 (-1) or -2. That is two successive multiplications by $\sqrt{-1}$ is the same as multiplication by -1 or effects a rotation of 180° and multiplication by $\sqrt{-1}$ then means rotation by one half of 180° or by 90°. Therefore $2\sqrt{-1}$ is the distance in the direction rotated 90° from the reference axis and such numbers as $2\sqrt{-1}$ are known as quadrature numbers, that is they represent distances not to the right or left of the reference point but upwards or downwards as the number is prefixed by the positive or negative sign. For convenience of use the $\sqrt{-1}$ is usually represented by the letter j and as such is used in most texts on alternating currents.

The quadrature number extends the field of operations from the line to the plane and by so doing makes possible the simple mathematical representation of the usual alternating current vectors.

As shown above the negative number has a physical meaning in case two opposite directions exist and in plane geometry where four directions exist the quadrature number is also real that is, it has a meaning, but is has no meaning if applied to problems in which only two directions exist.

By the use of positive and negative numbers all

points in a line could be represented numerically as distances from a chosen point O. By the use of the quadrature number all points in the entire plane can now be represented as distances from the coordinate axis X and Y. Thus in figure 21 any point P of the plane has a horizontal distance from the Y axis = O A = +3 and a vertical distance from the X axis = A P = +3j. Expressing O A and A P as vectors the location of the point P relative to the origin is $\overline{OA} + j\,\overline{AP}$ or $P = 3 + j3$.

The General Number

Such a combination of an ordinary number and a quadrature number is called a general number or a complex quantity. Complex quantities are especially convenient in alternating current work since by their use points may be very readily located at any position in a plane as is found necessary in almost every vector diagram of alternating current circuits.

Representation of General Numbers

General numbers are usually represented by capitals and their components, the ordinary number and the quadrature number by small letters. The distance of the point, which represents the general number, from the coordinate center is called the absolute value or the real value and is usually designated by a small letter, but when represented by a large letter it is distinguished from the general number by using for the latter a different kind of type or by placing a dot

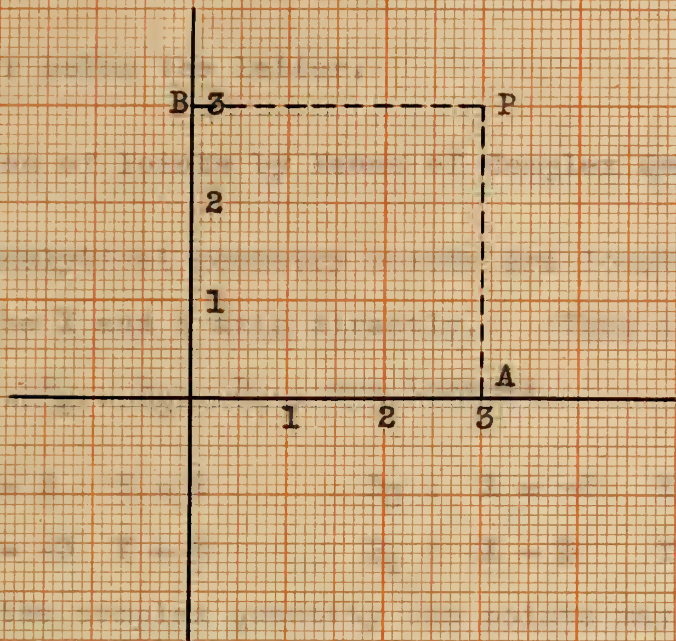

Fig. 21

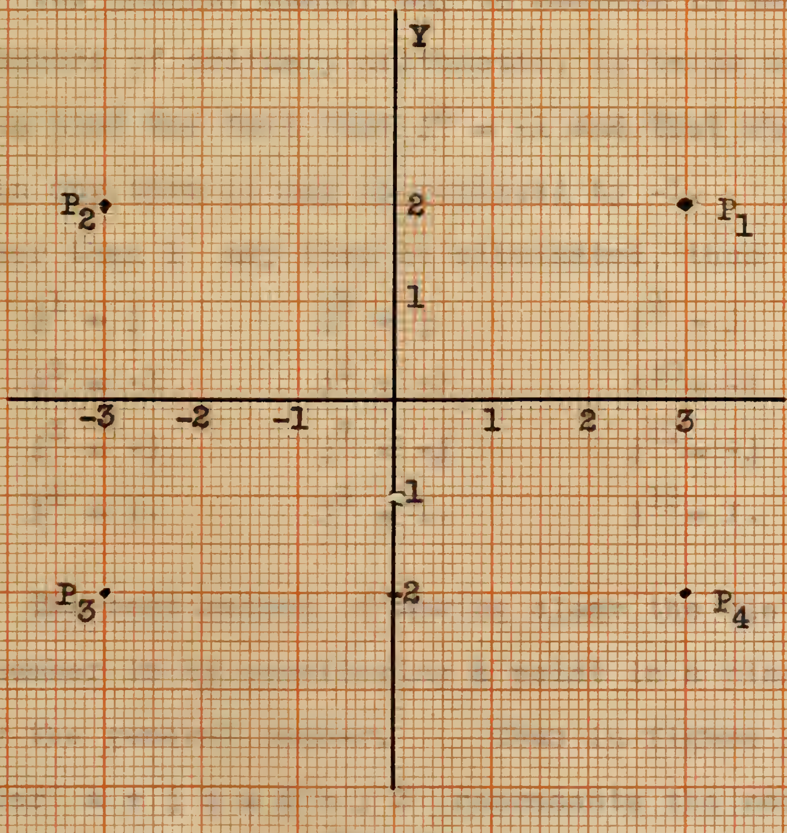

Fig. 22

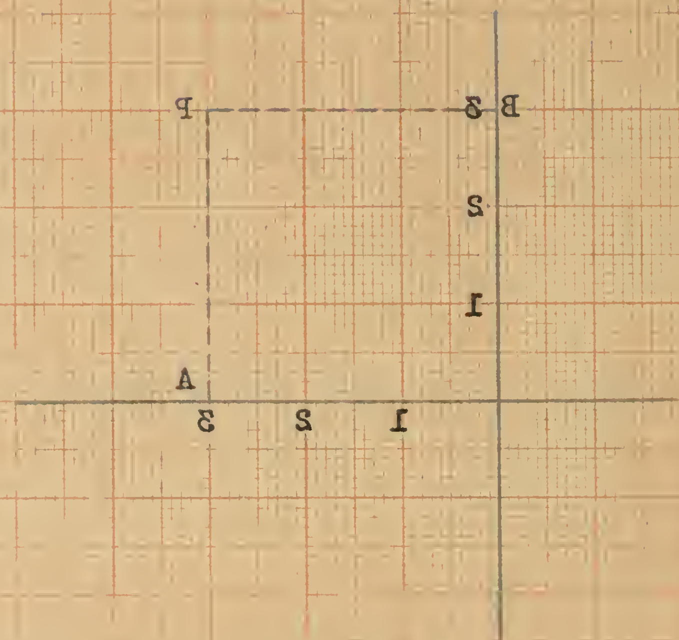

Fig. 21

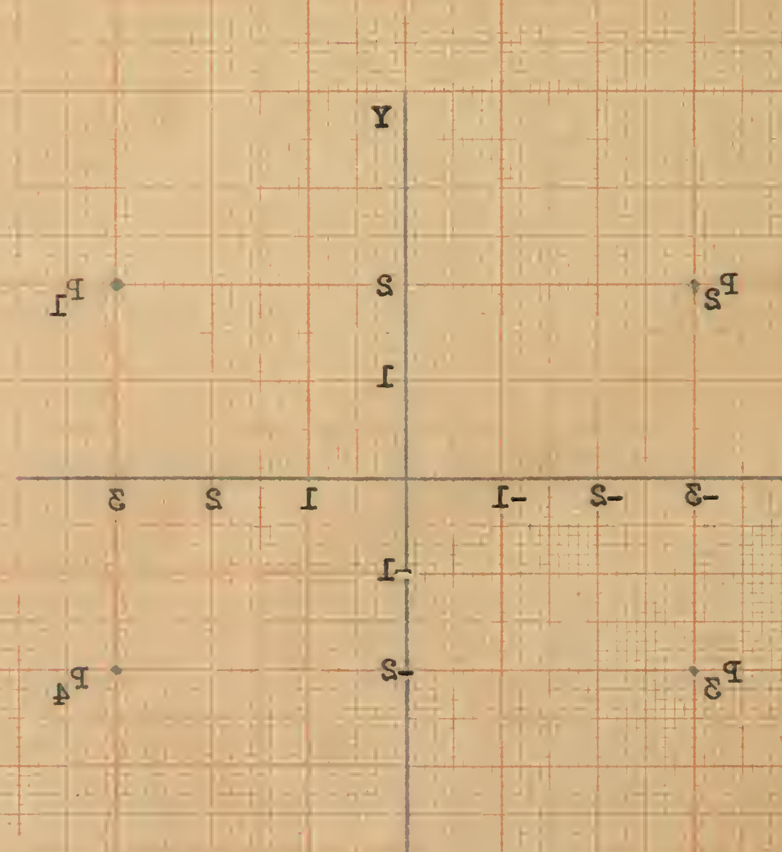

Fig. 22

either above or below the letter.

Representation of Points by Means of Complex Quantities

In analytical geometry points are located by reference to the X and Y axis directly. Thus in figure 22 the points P_1 , P_2 , P_3 , P_4 , are located

P_1 : X = 3 Y = 2 $\qquad$ P_3 : X = -3 Y = -2

P_2 : X = -3 Y = 2 $\qquad$ P_4 : X = 3 Y = -2

By the use of the complex quantity the points may be located by,

$$P_1 = 3 + j\,2 \qquad P_3 = -3 - j\,2$$

$$P_2 = -3 + j\,2 \qquad P_4 = 3 - j\,2$$

The general number may be handled in the same manner as the numbers of ordinary arithmetic, it being only necessary to keep in mind the fact that $j^2 = -1$ and that whenever j^2 appears in any term it may be replaced by -1. All powers of j higher than 1 may then be eliminated, thus

$$j^1 = j \qquad j^5 = j \qquad j^9 = j$$

$$j^2 = -1 \qquad j^6 = -1 \qquad j^{10} = -1$$

$$j^3 = -j \qquad j^7 = -j \qquad j^{11} = -j$$

$$j^4 = 1. \qquad j^8 = 1. \qquad j^{12} = 1.$$

The best method of making clear the use of the general number is by considering a point in a plane as represented by the general number. Thus in figure 23 the general number $a + j\,b = 4 + j\,3$ represents the point P which has a horizontal distance from the Y axis $= \overline{O\,A} = \overline{B\,P} = 4 = a$

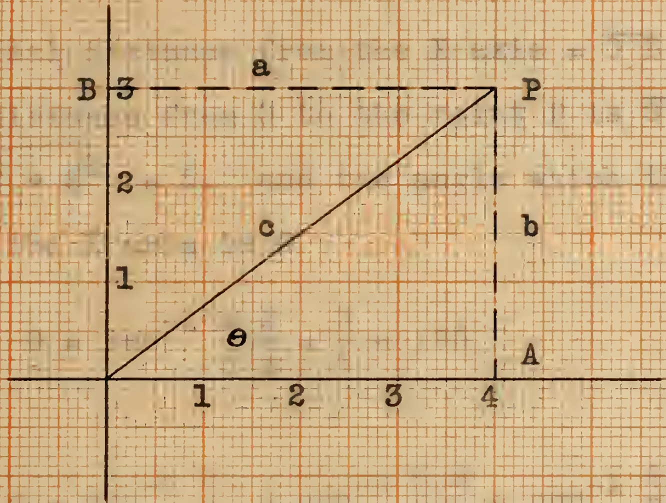

Fig. 23

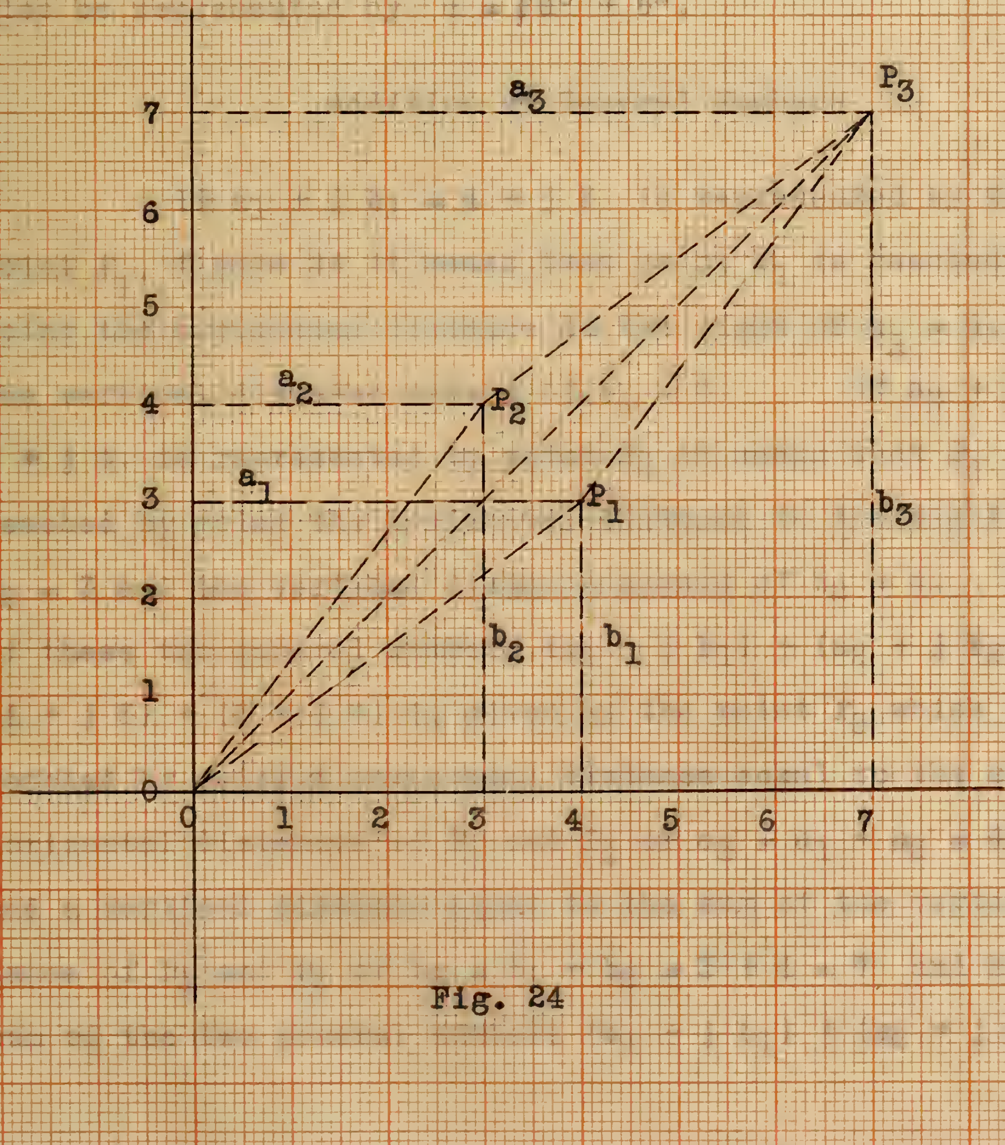

Fig. 24

Fig. 23

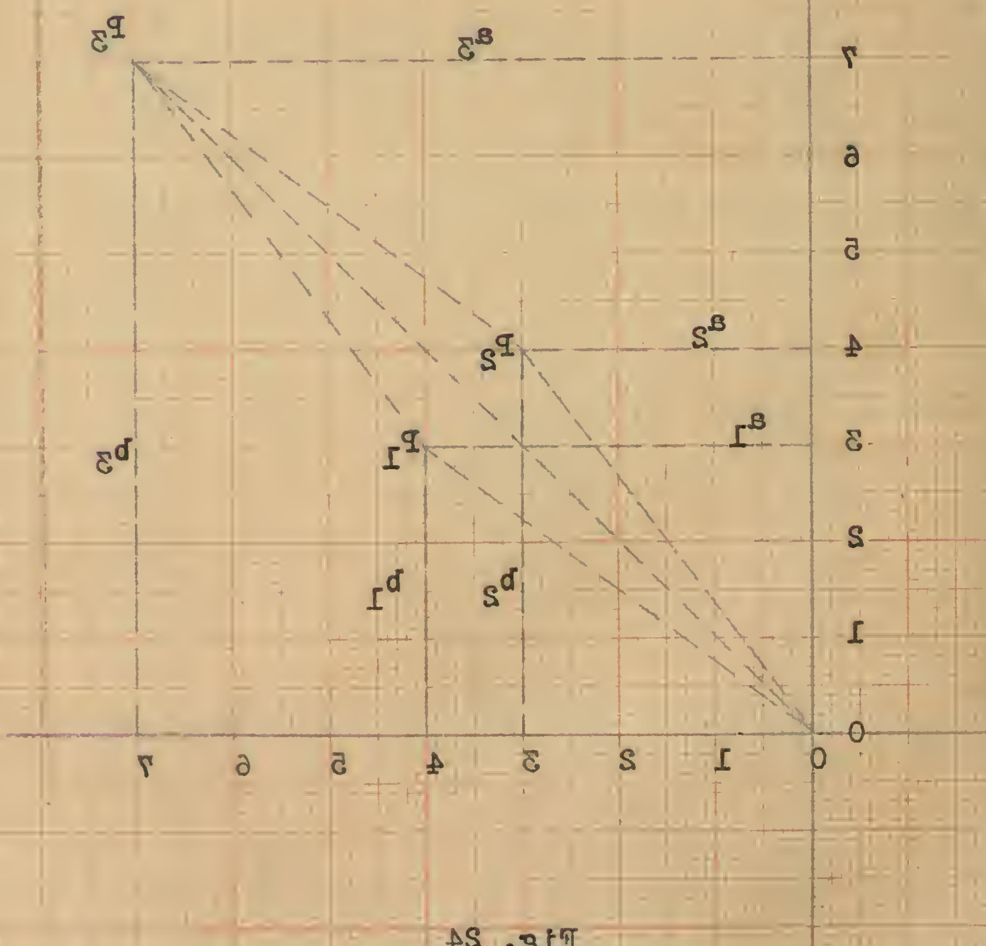

Fig. 24

and a vertical distance from the X axis $= \overline{OB} = AP = b = 3.$ The total distance from 0 to the point P is $\overline{OP} = \sqrt{\overline{OA}^2 + \overline{AP}^2}$ $= \sqrt{4^2 + 3^2} = 5.$ and the angle which the line O P makes with the X axis is θ

$$\theta = \tan^{-1} \frac{AP}{OA} = \frac{3}{4} = .75$$

or

$$\theta = \cos^{-1} \frac{a}{c} = \cos^{-1} \frac{OA}{c} = \cos^{-1} \frac{4.}{5.} = \cos^{-1} .8$$

The general number is here represented by $a + j\,b$ but it may also be represented by $c = \sqrt{a^2 + b^2}$.

Addition of General Numbers

If $a_1 + j\,b_1 = 4 + j\,3$ is represented by the point P_1, figure 24 it means that point P_1 is reached by going the horizontal distance to the right of $a_1 = 4.$ and the vertical distance upward of $b_1 = 3.$ If $a_2 + j\,b_2 = 3 + j\,4.$ is represented by point P_2 it means that P_2 is reached by going the horizontal distance to the right of $a_2 = 3$ and the vertical distance upward of $b_2 = 4.$ The sum of these two general numbers $(a_1 + j\,b_1) + (a_2 + j\,b_2) = (4 + j\,3) + (3 + j\,4)$ is given by the point P_3 which is reached by going a horizontal distance equal to the sum of the horizontal distances of P_1 and P_2 or $a_3 = a_1 + a_2 = 4 + 3 = 7.$, and a vertical distance equal to the sum of the vertical distance of P_1 and P_2 or $b_3 = b_1 + b_2 = 3 + 4 = 7$ and hence the sum of the two general numbers $(a_1 + j\,b_1) + (a_2 + j\,b_2)$ is

given by the general number $(a_3 + j\ b_3)$ or

$$(a_3 + j\ b_3) = (a_1 + j\ b_1) + (a_2 + j\ b_2) = 7 + j\ 7.$$

Geometrically, point P_3 is derived from points P_1 and P_2 by the diagonal $\overline{O\ P_3}$ of the parallelogram $O\ P_1\ P_3\ P_2$ constructed with $\overline{O\ P_1}$ and $\overline{O\ P_2}$ as sides as shown in figure 24. Stated literally, the addition of general numbers represents geometrical combination by the parallelogram law.

Subtraction of General Numbers

In figure 24, P_3 represents the number

$$a_3 + j\ b_3 = 7 + j\ 7$$

and P_1 represents the number

$$a_1 + j\ b_1 = 4 + j\ 3$$

The difference of these numbers is represented by a point P_2 which is reached by going the respective differences of the horizontal distances and of the vertical distances of the point P_3 and P_1. P_2 is then represented by

$$a_2 = a_3 - a_1 = 7 - 4 = 3$$

and

$$b_2 = b_3 - b_1 = 7 - 3 = 4.$$

Therefore the difference of the two general numbers $(a_3 + j\ b_3)$ and $(a_1 + j\ b_1)$ is given by the general number

$$\begin{aligned}(a_2 + j\ b_2) &= (a_3 + j\ b_3) - (a_1 + j\ b_1)\\ &= (a_3 - a_1) + j\ (b_3 - b_1) = 3 + j\ 4.\end{aligned}$$

This difference $a_2 + j\ b_2$ is represented by one side of the

parallelogram $O\ P_1\ P_3\ P_2$, figure 24, which has $\overline{O\ P_1}$ for one side and $O\ P_3$ for the diagonal. Subtraction of general numbers, thus geometrically represents the resolution of a vector, $\overline{O\ P_3}$, figure 24 for example, into two components $\overline{O\ P_1}$ and $O\ P_2$, by the parallelogram law.

The foregoing shows the principal advantage of the use of the complex quantity in engineering calculations. If the vectors are represented by complex quantities combination and resolution of vectors is carried out by the simple addition or subtraction of their general numerical values.

Multiplication of General Numbers

If $\underset{\bullet}{A} = a_1 + j\ b_1$ and $\underset{\bullet}{B} = a_2 + j\ b_2$ are two general numbers their product is obtained by simple multiplication in which the only possible difficulty is in the manipulation of the j terms. Thus using the numbers given above

$$\begin{aligned} \underset{\bullet}{A}\ \underset{\bullet}{B} &= (a_1 + j\ b_1)\ (a_2 + j\ b_2) \\ &= a_1\ a_2 + j\ a_1\ b_2 + j\ a_2\ b_1 + j^2\ b_1\ b_2 \\ &= (a_1\ a_2 - b_1\ b_2) + j\ (a_1\ b_2 + a_2\ b_1). \end{aligned}$$

Since, as must be remembered, $j^2 = -1$ this is nothing more difficult than the simplest algebraic multiplication.

The multiplication of complex quantities, like subtraction, can not always be carried out in a physical sense. That is, there are certain conditions which must be fulfilled in order that the operation may have a physical meaning. Thus, on a vector diagram, a current may be represented by

$$\underset{\bullet}{I} = i + j\ i'$$

and an impedance may be represented by

$$\underset{\cdot}{Z} = r + j\,x$$

If the two are multiplied together there results a voltage

$$\underset{\cdot}{E} = \underset{\cdot}{I}\,\underset{\cdot}{Z} = (i + j\,i')\,(r + j\,x)$$

$$= i\,r - i\,x + j(i\,x + i'\,r)$$

This voltage is real and has the same frequency as the current and hence may be represented on the same diagram.

If it should be desired to show the power consumed in the circuit, that is the product of $\underset{\cdot}{E}$ and $\underset{\cdot}{I}$ it would be found that the result is wrong physically. Upon considering the nature of the quantities it is seen that this should be expected as it has been previously shown that the power fluctuates at a frequency of twice that of the current and voltage and therefore can not be represented on the same vector diagram.

Division of the General Number

In the alternating current work it is very important to be able to find the current flowing in a circuit if the impedance and voltage are expressed in complex quantities.

In general the current flowing is found by dividing the e. m. f. by the impedance of the circuit and if the complex quantities are to be used it must be feasible to divide them by each other as well as to add and multiply them.

Suppose the e. m. f. impressed upon a circuit is $\underset{\cdot}{E} = e + j\,e'$ and that the impedance of the circuit is $\underset{\cdot}{Z} = r + j\,x$, then the current flowing is

$$\dot{I} = \frac{\dot{E}}{\dot{Z}} = \frac{e + j\,e'}{r + j\,x}$$

If now the denominator is multiplied by its conjugate function, in this case $(r - j\,x)$, the work will be very much simplified and the calculation of the current made readily possible. Then

$$I = \frac{(e + j\,e')\,(r - j\,x)}{(r + j\,x)\,(r - j\,x)} = \frac{(e + j\,e')\,(r - jx)}{r^2 + x^2}$$

$$= \frac{e\,r + e'\,x + j\,e'\,r - j\,e\,x}{r^2 + x^2}$$

$$= \frac{e\,r + e'\,x}{Z^2} + j\,\frac{e'\,r - e\,x}{Z^2}$$

from which the value of the current may very readily be calculated for any numerical case.

If desired the quadrature term may be eliminated from the numerator instead of from the denominator but in most alternating current problems the method illustrated above is used.

Familiarity with the use of the complex quantity is very desirable and since this may best be obtained by practice a number of problems are herewith given.

Problem: Given E = 1000 volts, r=10 ohms, inductive reactance x_L= 20 ohms, capacity reactance x_c= 5 ohms. Find the complex and also the real value of the current flowing and the power factor for the total circuit.

$$Z = \sqrt{r^2 + (x_L - x_c)^2} = \sqrt{10^2 + 15^2} = 18.$$

$$I = \frac{E}{Z} = \frac{1000}{18} = 55.5 \text{ amp. (Real value)}$$

Also for the complex expression for the current

$$\underset{\cdot}{I} = \frac{E}{r + j\,x} = \frac{1000}{10 + j\,15}$$

$$= \frac{1000\,(10 + j\,15)}{10^2 + 15^2}$$

$$= \frac{10000}{325} - j\,\frac{15000}{325}$$

$$\underset{\cdot}{I} = 30.7 - j\,46.2$$

and

$$\cos\theta = \frac{30.7}{55.5} = .553 \text{ or } \theta = 56\ \ 25'.$$

The above complex expression of current shows further that if the e. m. f. is taken as the reference vector, that is the horizontal line in the vector diagram, the current vector will fall in the fourth quadrant, that is, it is behind the e. m. f. in phase and is known as a lagging current.

In the above problem the e. m. f. was expressed as a real number but the problem would have been equally simple had E been expressed in complex form.

In general the vectors on a diagram are referred to the horizontal as a reference axis and when not otherwise stated this is understood to be the case. Thus in the above problem it was given that E = 1000. It is then understood that this E is to be the reference vector.

Any one of the vectors representing a given circuit might be used as the reference vector but it is most common and also more simple to use the current as the reference vector in series circuits and the e. m. f. as reference vector

in parallel circuits.

Problem: Given E = 900 + 800 j, r = 50 ohms, x_L = 60 ohms, x_c = 80 ohms and Z = r + j x - j x_c = 50 - j 20, to find the current flowing in the circuit.

$$\dot{I} = \frac{\dot{E}}{\dot{Z}} = \frac{900 + j\,800}{50 - j\,20} = \frac{(900 + j\,800)\,(50 + j\,20)}{50^2 + 20^2}$$

$$= \frac{900 \times 50 + j\,900 \times 20 + j\,800 \times 50 + 800 \times 20\,j2}{2500 + 400}$$

$$= 10 + j\,20$$

$$= 22.4 \text{ Amp. (Real value)}$$

This is represented on the vector diagram in figure 25. The current here is leading the e. m. f. by some angle θ due to the capacity reactance in the circuit.

Problem: Given a parallel circuit, as represented in figure 26, having an impressed e. m. f. of E = 100 volts. To find the current and phase angle of each branch of the circuit and also the current and phase angle of the circuit as a whole and plot the vector diagram using the impressed e. m. f. as a reference vector.

For branch #1.

$$\dot{I}_1 = \frac{E}{Z_1} = \frac{100}{r + j\,7} = \frac{100\,(2 - j\,7)}{2^2 + 7^2}$$

$$= \frac{200}{53} - j\,\frac{700}{53}$$

$$\dot{I}_1 = 3.77 - j\,13.2$$

$$= 13.72 \text{ Amp. (real value)}$$

For branch #2.

$$I_2 = \frac{E}{Z_2} = \frac{100}{5 - j\,2} = \frac{100\,(5 + j\,2)}{5^2 + 2^2}$$

$$= \frac{500}{29} + j\,\frac{200}{29}$$

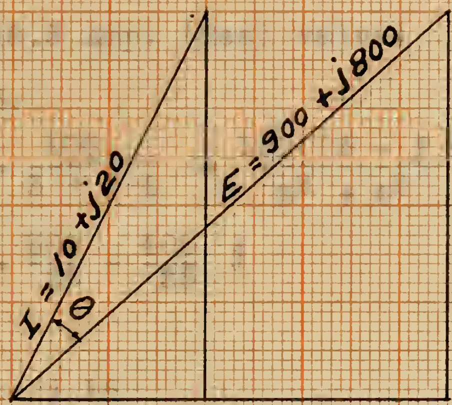

Fig. 25

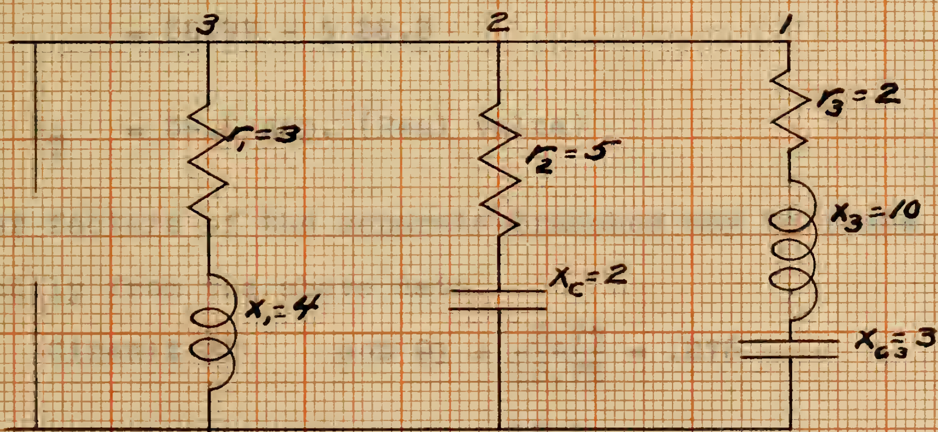

Fig. 26

Fig. 25

Fig. 26

$$\dot{I}_2 = 17.2 + j\,6.9$$

$$= 18.5 \text{ Amp. (Real value)}$$

For branch #3.

$$\dot{I}_3 = \frac{E}{Z_3} = \frac{100}{3 + j\,4} = \frac{100\,(3 - j\,4)}{3^2 + 4^2}$$

$$= \frac{300}{25} - \frac{400}{25}\,j$$

$$\dot{I}_3 = 12 - j\,16$$

$$= 20 \text{ Amp. (Real value)}$$

The total current is then found by the addition of the several branch currents by simple addition of the general numbers as before suggested or

$$\dot{I}_T = \dot{I}_1 + \dot{I}_2 + \dot{I}_3$$

$$= (3.77 - j\,13.2) + (17.2 + j\,6.9) + (12 - j\,16)$$

$$= 32.97 - j\,22.3$$

$$I_T = 39.8 \text{ Amp. (Real value)}$$

The power factors of the separate branches may be found very readily from the above data.

Circuit #1 $\quad \cos\theta_1 = \frac{3.77}{13.72} = .275$

Circuit #2 $\quad \cos\theta_2 = \frac{17.2}{18.5} = .93$

Circuit #3 $\quad \cos\theta_3 = \frac{12}{20} = .60$

Total Circuit $\quad \cos\theta = \frac{32.97}{39.8} = .827$

The vector diagram of this circuit is shown in figure 27.

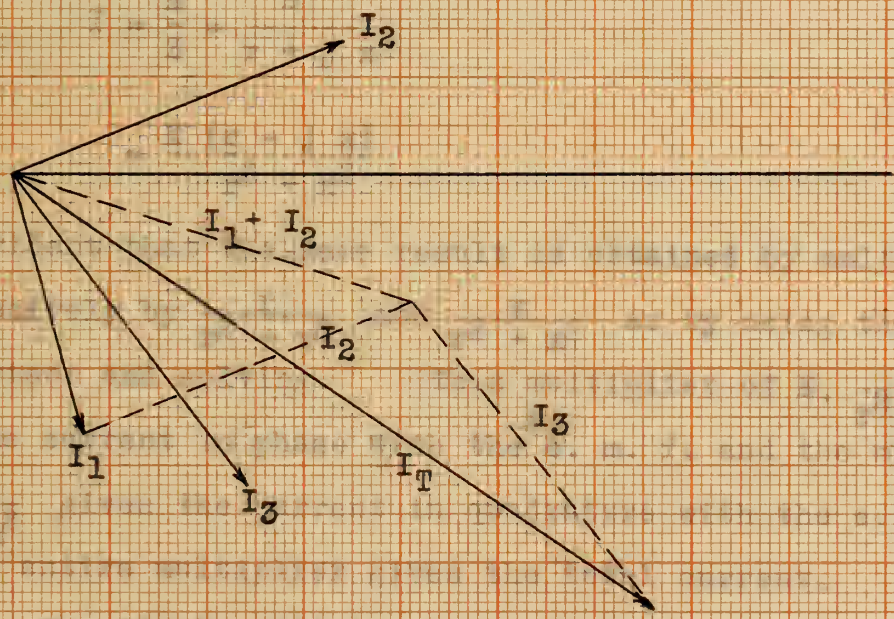

Fig. 27
Vector diagram for problem on page 82.

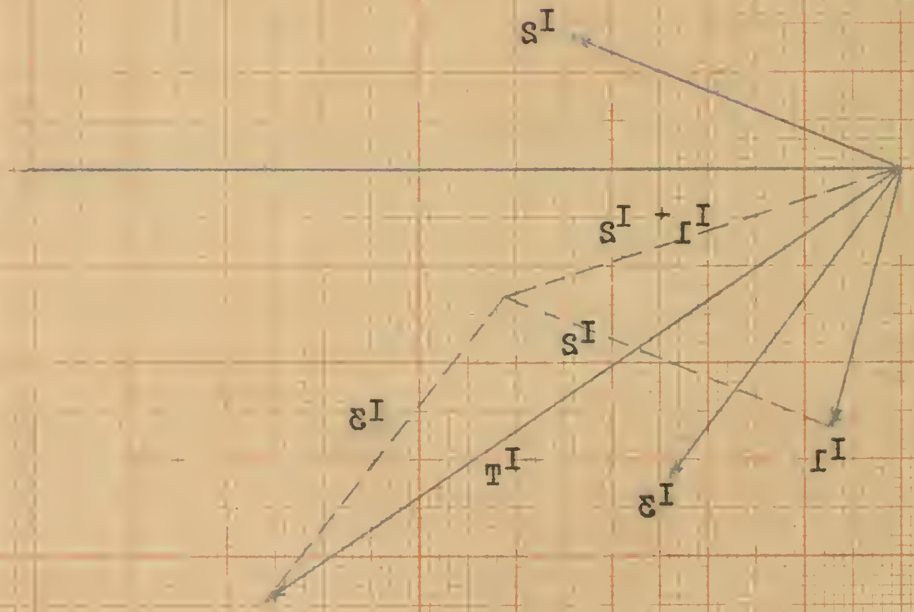

Fig. 27

Vector diagram for problem on page 82.

Conductance, Susceptance and Admittance

The above method of solution of parallel circuits has the advantage for the beginner that it gives the complex values of the currents directly for each circuit but after one has become familiar with the operation it is often advisable, especially in complicated circuits to refrain from solving for the current directly and get only the constants of the various branches.

Thus

$$I = \frac{E}{Z} = \frac{E}{r + j\,x}$$

$$= \frac{E\,(r - j\,x)}{r^2 + x^2}$$

It is evident that the same result is obtained by multiplying E successively by $\frac{r}{r^2 + x^2}$ and $\frac{x}{r^2 + x^2}$ as by using the $r + j\,x$ as a divisor and solving. This multiplier of E, $\frac{r}{r^2 + x^2}$, gives the current in phase with the e. m. f. and the multiplier $j\,\frac{x}{r^2 + x^2}$ gives the current in quadrature with the e. m. f. and the entire multiplier gives the total current.

The first multiplier above is called the conductance and is represented by the letter g. The second multiplier is called the susceptance and is represented by the letter b and the total multiplier is called the admittance and is represented by the letter Y. Thus

Conductance $= g = \frac{r}{r^2 + x^2}$ (In phase component of current.)

Susceptance $= b = \frac{x}{r^2 + x^2}$ (Quadrature component of current.)

Admittance $= Y = g - j\ b$ (Total current)

$= \sqrt{g^2 + b^2}$ (Real value)

and since $E\ Y = I \quad Y = \frac{1}{Z}$.

In a problem such as the one just shown the manipulation of the conductance, susceptance and admittance would be as follows.

For Circuit #1.

$$Y_1 = \frac{1}{Z_1} = \frac{1}{2 + j\ 7} = \frac{2 - j\ 7}{.53} = .0377 - j\ .132$$

For Circuit #2.

$$Y_2 = \frac{1}{Z_2} = \frac{1}{5 - j\ 2} = \frac{5 + j\ 2}{29} = .172 + j\ .069$$

For Circuit #3.

$$Y_3 = \frac{1}{Z_3} = \frac{1}{3 + j\ 4} = \frac{3 - j\ 4}{25} = .12 - j\ .16$$

$$Y_T = Y_1 + Y_2 + Y_3 = .3297 - j\ .223$$

From which $\dot{I} = E\ Y$

$= 100\ (.3297 - j\ .223)$

$= 32.97 - j\ 22.3$

VII

THE TRANSMISSION LINE

The transmission line is in general nothing more than an electric circuit which may contain resistance and inductive or capacity reactance or a combination of all three. These impedances are represented as connected in the circuit between the generator and the load. The resistance and inductive reactance are considered as connected in series with the load and the capacity reactance in multiple therewith.

Impedance of Transmission Lines

That these impedances are real and not assumed is readily seen upon investigation. Any wire no matter how large its cross-section has an ohmic resistance and hence a transmission line must have such a resistance.

Similarly any wire carrying a current is surrounded by a flux and the variation of this flux through the space between one wire and the return conductor sets up an e. m. f. of self inductance in the wire. That is, the action of the magnetic flux sets up an e. m. f. opposing the e. m. f. which is sending the current and hence the wire is said to have self inductance.

Between the wires of a transmission line or between the wires and ground there is set up an electrostatic field. Since the field is constantly varying with the current in the

wires its charge is varying, that is it is acting like a condenser. The effect in a long high voltage transmission line is the same as if condensers were connected from wire to wire or from wire to ground at various points along the line.

Problem: A transmission line delivers 500 K. W. at 20,000 volts over a line of 75 ohms resistance and 100 ohms inductive reactance. Find the voltage at the generator $\dot{E}_o$ and the power factor at the generator when the load is 500 K. W. at P. F. of 1, .8 lag and .8 lead respectively. Plot the vector diagram for each case, see figures 28, 29, 30.

Taking E, the load voltage, as reference vector the expression for generator voltage may be written as

$$E_o = E + \dot{I}\,\dot{Z}$$
$$= E + (i + j\,i')(r + j\,x)$$
$$= E + i\,r - i'\,x + j\,(i\,x + i'\,r)$$

If then the values of i and i' (the in phase and quadrature components of the current) are known the value of E_o may very readily be calculated.

For P. F. = 1.

$P = 500{,}000$ watts.

$E = 20{,}000$ volts.

$i = \dfrac{500000}{20000} = 25$ amp.

$i' = 0$ Since P. F. = unity.

For P. F. = .8 lead or lag, i is the same since the power and voltage are constants.

$$I = \frac{i}{.8} = \frac{i'}{.6}$$

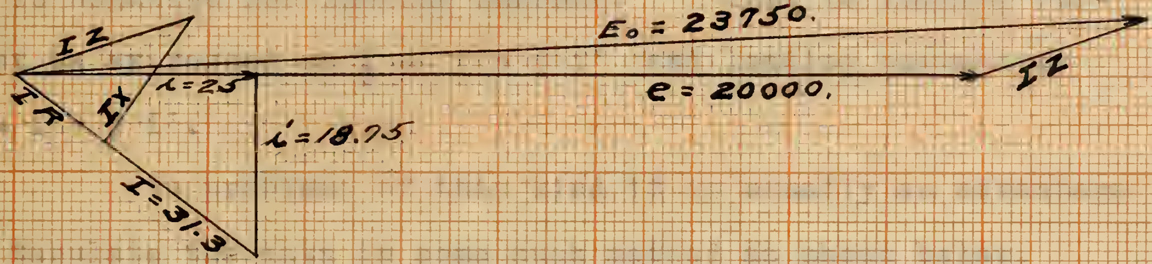

Fig. 28

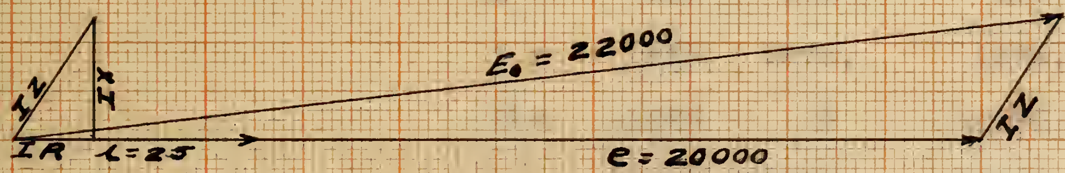

Fig. 29

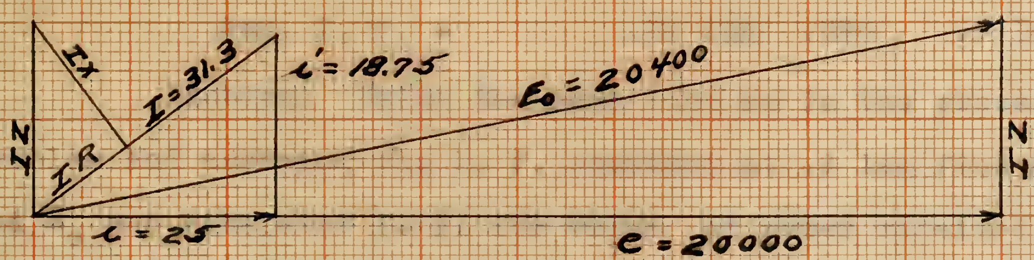

Fig. 30

$i' = 18.75$ amp. for .8 lag or .8 lead, except that i' is negative for lagging current. That is lagging current is expressed as $\dot{I} = i - j\,i'$ and leading current as $\dot{I} = i + j\,i'$.

In problems of this kind it is usually an advantage to tabulate the data in some such method as shown below.

P. F.	.8 Lag	1.	.8 Lead
i	25	25	25
i'	-18.75	0	+18.75
r	75	75	75
x	100	100	100
$i\,r$	1875	1875.	1875.
$i'x$	-1875	0	+1875
e	20000	20000	20000
(a) $e + i\,r - i'x$	23750	21875	20000
$i\,x$	2500	2500	2500
$i'r$	-1405	0	+1405
(b) $j\,(i\,x + i'r)$	1095	2500	3905
$\dot{E}_o = (a + j\,b)$	----	----	----
E	23750	22000	20400
Gen. P. F. =	.72 Lag	.99+	.9 Lead

Problem: Using the same line as in the preceding problem and assuming the P. F. constant at .9 lag find $\dot{E}_o$, and E_o and generator P. F. for value of power received by load of 0, 100 K. W., 200 K. W., 400 K. W., 800 K. W.

P = K.W.	0	100	200	400	800
P. F.	.9	.9	.9	.9	.9
i	0	5	10	20	40
i'	0	-2.41	-4.8	-9.6	-19.2
r	75	75	75	75	75
x	100	100	100	100	100
i r	0	375	750	1500	3000
i'x	0	-241	-480	-960	-1920
e	20000	20000	20000	20000	20000
(a) e + i r - i'x	20000	20616	21230	22460	24920
i x	0	500	1000	2000	4000
i' r	0	182	364	728	1456
(b) j (i x + i'r)	0	682	1364	2728	5456
E_o (a + j b)	20000	20616 + j682	21230 + j1364	22460 +j2728	24926 +j5456
E_o	20000	20620	21230	22500	25010
Tan α	0	.006	.03	.057	.101
α	0	16'	1° 45'	3° 15'	6°
β	-25° 45'	-25° 45'	-25° 45'	25° 45'	25° 45'
θ	25° 45'	26° 1'	27° 30'	29° 0'	31° 45'
cos θ = Gen. P. F.	.9	894	.885	.872	.851

In the problems just considered the generator power factor has been calculated by the relation of the true power to the product of generator e. m. f. and generator voltage. The true power in these cases is calculated by adding to the power delivered to the load the I^2r loss of the line. This method gives the correct result and may be used but it requires the finding of the real value of both the generator

voltage and current and this is a rather monotonous preceeding when done over and over again.

Power by Complex Quantities

Consider two vectors E and I, figure 31, which are displaced from each other by some angle θ. The power may be calculated by the use of the complex quantity thus

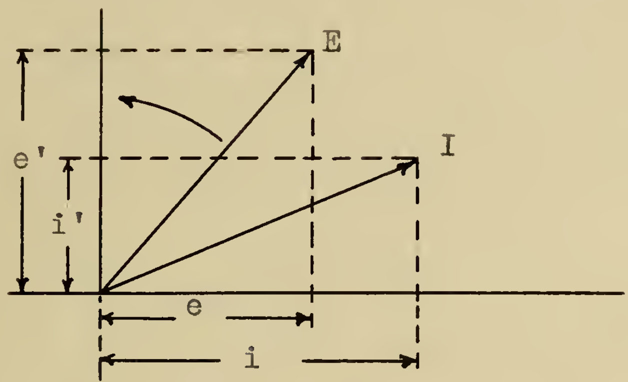

Figure 31

Here $E = e + j\,e'$ and $I = i + j\,i'$. Resolving the vectors into their components along the line of the axes of coordinates it is seen that the quadrature components of E and I are in phase as are also the "in phase" components. Since a current times an e. m. f. times the cosine of the phase angle gives power, the power represented by these components of current and voltage is as follows.

Power $= e\,i \cos 0^\circ + e'\,i' \cos 0^\circ + e'\,i \cos 90^\circ + e\,i' \cos 90^\circ$, which gives, when the values of the cosines are substituted

Power $= e\,i + e'\,i'$.

Thus it is a very simple proceeding to obtain the power in a circuit when the current and voltage are expressed

as complex quantities. The value of the connecting sign can readily be determined from the expressions for current and voltage.

Maximum Value of Power That May be Transmitted Over a Given Line Having Resistance and Inductive Reactance

In this case it is well to assume the generator voltage E_0 as constant and this is in general the case as it is evident that if a station has several transmission lines leading from it the voltage could not be changed to accomodate the receiving voltage of each separate line. Assume further that the load has resistance and reactance represented by $Z_1 = (r_1 + j\,x_1)$ and line impedance of $r + j\,x$. The current which would flow under these conditions is

$$\dot{I} = \frac{E_0}{\dot{Z}} = \frac{E_0}{(r + jx) + (r_1 + jx_1)} = \frac{E_0\left[(r + r_1) - j\,(x + x_1)\right]}{(r + r_1)^2 + (x + x_1)^2}$$

Let

$$a = \frac{(r + r_1)}{(r + r_1)^2 + (x + x_1)^2}$$

$$b = \frac{(x + x_1)}{(r + r_1)^2 + (x + x_1)^2}$$

Then

$$\dot{I} = E_0\,(a - j\,b)$$

In order to get the power delivered it is first necessary to obtain the voltage at the load. Calling this voltage E_1,

$$E_1 = I\,Z_1 = E(a - j\,b)\,(r_1 + j\,x_1)$$

$$= E\left[(a\,r_1 + b\,x_1) + j\,(a\,x_1 + b\,r_1)\right]$$

$$= E\,(c + j\,d)$$

$$c = (a\,r_1 - b\,x_1)$$

$$d = (a\, x_1 - b\, r_1)$$

The power under these conditions would then be

$$P = e\, i + e'\, i'$$

or

$$P = E^2\, (a\, c - b\, d)$$

Substituting the values of a, c, b and d and clearing the fractions

$$P = \frac{E^2\, r_1}{(r + r_1)^2 + (x + x_1)^2} = \frac{E^2\, r_1}{Z^2}$$

$$\frac{\partial P}{\partial r_1} = \frac{E^2\, Z^2 - E^2 r_1\, (2\, r + 2\, r_1)}{Z^4}$$

$$\frac{\partial P}{\partial x_1} = \frac{-E^2\, r_1\, (2\, x + 2\, x_1)}{Z^4}$$

Equating the above partial derivatives to zero and solving it is found that

$$x = -x_1$$

$$r = r_1$$

Substituting in the condition

$$\frac{\partial^2 P}{\partial x^2_1} \cdot \frac{\partial^2 P}{\partial r^2_1} > \left(\frac{\partial^2 P}{\partial x_1\, \partial r_1} \right)^2$$

It is found to hold for these values of r_1 and x_1 and since $\frac{\partial^2 P}{\partial r^2_1}$ is negative the values as found give the maximum power.

The condition for maximum power is then that the resistances are equal and that the reactance of the load is negative to that of the line, that is if the line reactance is inductive the load reactance must be capacity or vice versa.

In a similar manner it may be proved that in case the load reactance is zero the maximum power is delivered when

$$r_1 = \sqrt{r^2 + x^2} .$$

Problem: Given a generator supplying power to a transmission line at constant voltage E = 1000. The impedance of the line is r + j x = 2.5 + j 6. Find the maximum power that may be transmitted if the receiving circuit is non inductive and plot the relation between power and receiving circuit resistance. See figure 32.

Tabulation for Maximum Power

r_1		1	3	5	6	8	12
r		2.5	2.5	2.5	2.5	2.5	2.5
$r + r_1$		3.5	5.5	7.5	8.5	10.5	14.5
$(r + r_1)^2$		12.3	30.3	56.5	72.2	110.4	200
$x_1 = 0$ $x_1 + x = x$		6.	6.	6.	6.	6.	6.
$(x + x_1)^2$		36	36.	36.	36.	36.	36.
$\frac{r + r_1}{(x+x_1)^2 + (r+r_1)^2}$	(a)	.0725	.083	.0812	.0785	0717	.0615
$\frac{x + x_1}{(r+r_1)^2+(x+x_1)^2}$	(b)	.1242	.0905	.0648	.0554	.0409	.0254
$(a\ r_1 + b\ x_1) =$	(c)	.0725	.249	.4060	.4710	.5736	.738
$(a\ x_1 - b\ r_1) =$	(d)	-.1242	-.2715	-.324	-.3324	-.3272	.304
a c		.00526	.0207	.0329	.037	.0471	.0424
b d		-0154	-.0246	-.021	-.0184	.0133	-0077
(a c - b d)		.02066	.0453	.0539	.0554	.0544	0501
E^2		1000	----	----	----	----	----
E^2(a c - b d) $P_{k.w.}$		20.66	45.3	53.0	55.4	54.4	50.1

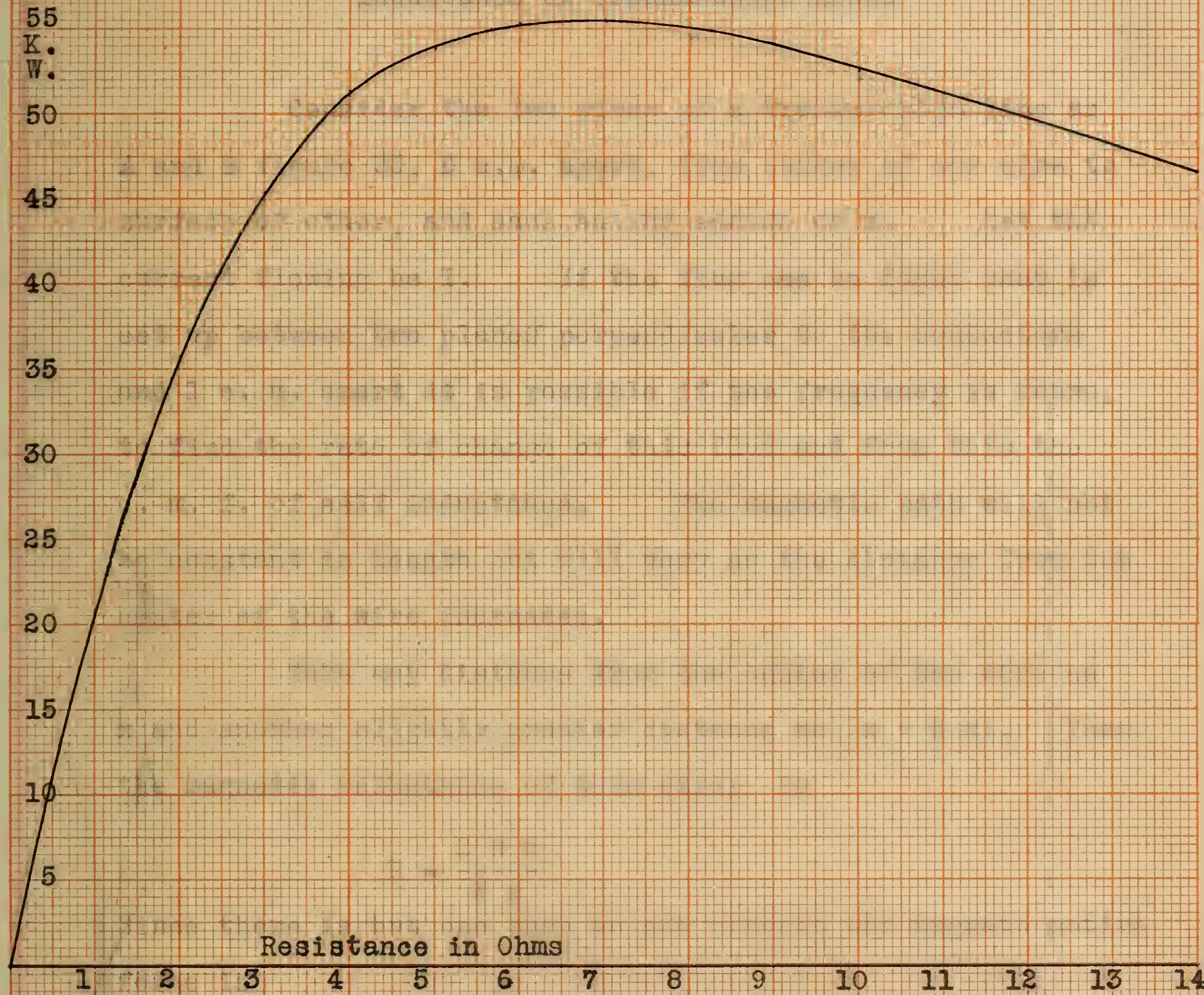

Fig. 32
Relation between power and receiving circuit resistance for a transmission line having

$R = 2 \quad X = 6 \quad E = 1000.$

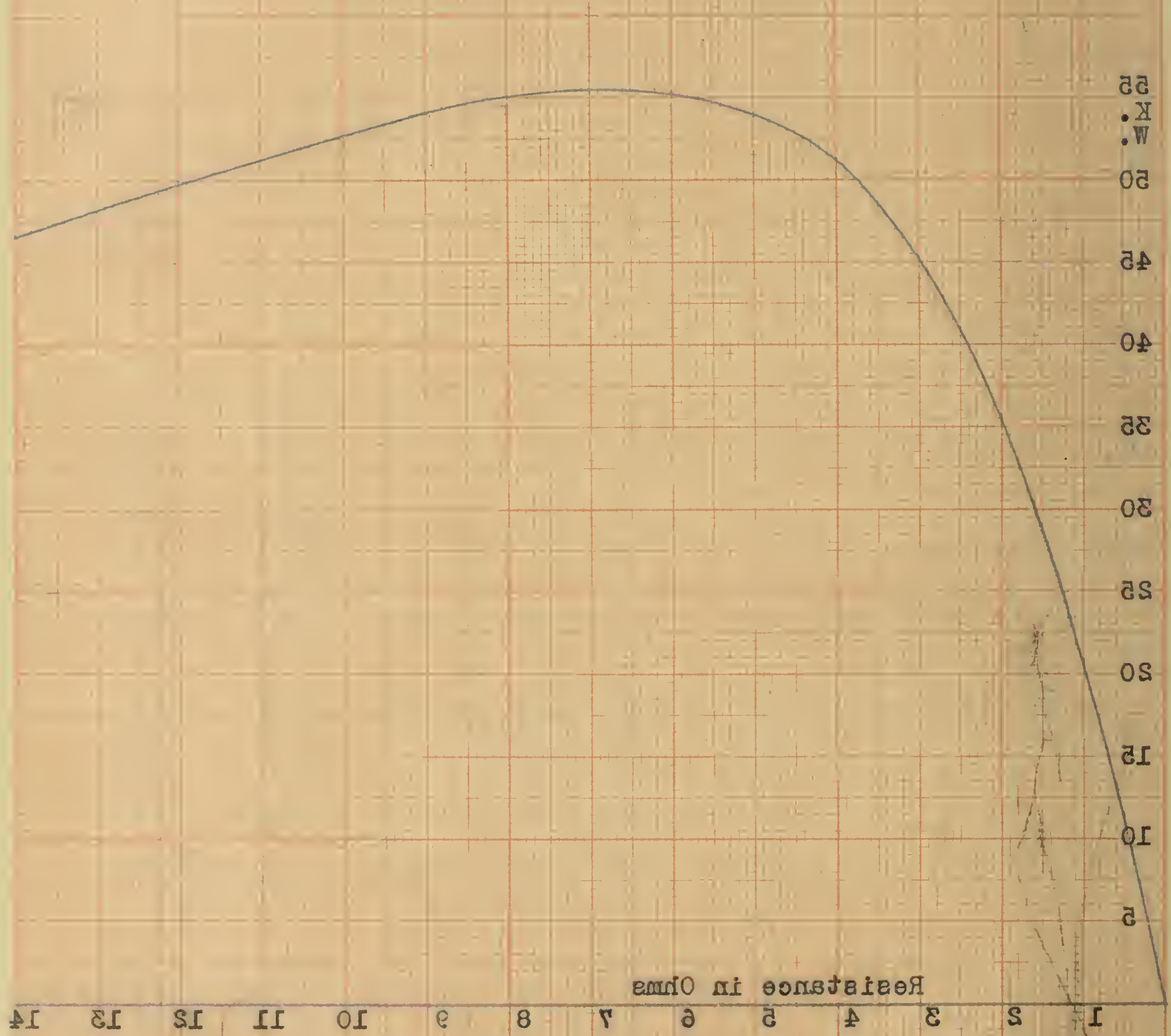

Fig. 32
Relation between power and receiving circuit
resistance for a transmission line having

R = 2   X = 6   E = 1000.

Inductance of Transmission Lines

Consider the two wires of a transmission line as A and B figure 33, D c.m. apart, from center of one wire to surface of other, and each having radius of r. Let the current flowing be I. If the flux can be found that is set up between two planes perpendicular to the conductors and 1 c. m. apart it is possible if the frequency is known, to find the rate of change of this flux and from this the e. m. f. of self inductance. The magnetic path will not be constant in length but will vary as the distance from the center of the wire increases.

Take any distance from the center of the wire as x and another slightly greater distance as (x + d x). Then the magnetic reluctance of this strip is

$$R = \frac{2 \pi x}{d x}$$

Since there is but one turn in the circuit the magneto motive force is

$$M.M.F. = .4 \pi I \text{ gilberts},$$

in which I is the current in practical units. Then the flux in this differential element of the space is

$$\phi \text{ or } d \phi = \frac{M.\ M.\ F.}{R} = \frac{.4 \pi I}{\frac{2 \pi x}{d x}} = \frac{.2 I d x}{x}$$

and the total flux in the space between the wires is

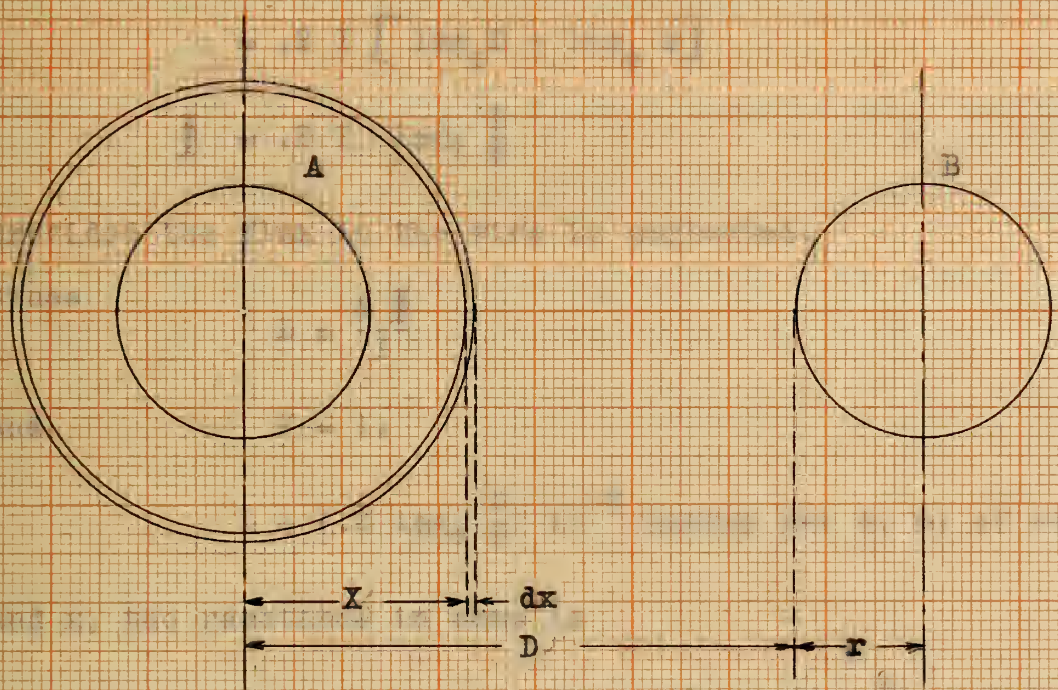

Fig. 33

Inductance between parallel cylindrical conductors.

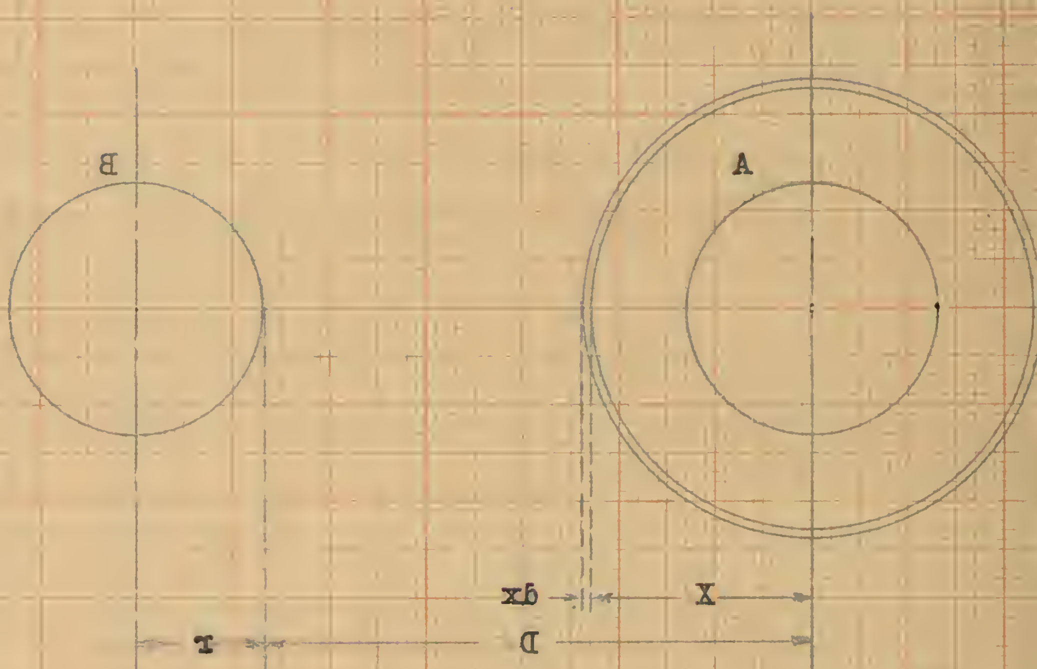

Fig. 33

Inductance between parallel cylindrical conductors.

$$\Phi = \int_r^D .2\ I\ \frac{dx}{x}$$

$$= .2\ I \log_e x + c\Big]_r^D$$

$$= .2\ I \left[\log_e D - \log_e r\right]$$

$$\Phi = .2\ I\ \log_e \frac{D}{r}$$

provided the flux in the wire is neglected.

Since $$L = \frac{N\ \Phi}{I}$$

and $$N = 1.$$

$$L = (.2 \log_e \frac{D}{r})\ 10^{-8} \text{ henrys per c. m. of wire.}$$

and x, the reactance in ohms is

$$x = 2\ \pi\ f\ L = 2\ \pi\ f\ 10^{-8}(.2 \log_e \frac{D}{r}) \text{ ohms per}$$

unit length.

From this L per mile of conductor is

$$L = .2 \log_e \frac{D}{r})\ \frac{5280 \times 12 \times 2.54}{10^8}$$

$$= \frac{1.61}{10^3}\ (2 \log_e \frac{D}{r}) \text{ Henrys per mile.}$$

Capacity of Transmissions Lines

The transmission lines thus far considered have had only resistance and inductive reactance but in reality they have also a capacity effect. That is, there is a certain

capacity between wires and also between wires and ground.

Consider two parallel cylindrical conductors as in figure 34, the capacity of which is determined as follows. Assume the two conductors A and B to be charged with a quantity of electricity Q. Conductor A positively charged +Q, and B negatively charged -Q. Each of these conductors is surrounded by a field of stress which gradually decreases to zero. The total field eminating from a unit length A is $4 \pi Q$ and hence the field intensity of an element at any distance x from A is

$$\frac{4 \pi Q}{2 \pi x} = \frac{2 Q}{x} .$$

Similarly the field intensity due to B is

$$- \frac{4 \pi Q}{2 \pi (D - x)} = - \frac{2 Q}{D - x} .$$

The resultant static field intensity is then

$$\frac{2 Q}{x} - \left(- \frac{2 Q}{D - x}\right) = 2 Q \left(\frac{1}{x} + \frac{1}{(D-x)}\right)$$

Consequently in moving the element from the plane of zero potential to the surface of the conductor its potential rises to

$$e = \int_{r}^{\frac{D}{2}} 2 Q \, dx \left(\frac{1}{x} + \frac{1}{D - x}\right)$$

which integrates to

$$e = 2 Q \log_e \frac{D - r}{r}$$

and since $Q = c\, e$ where C = capacity and e = e. m. f.

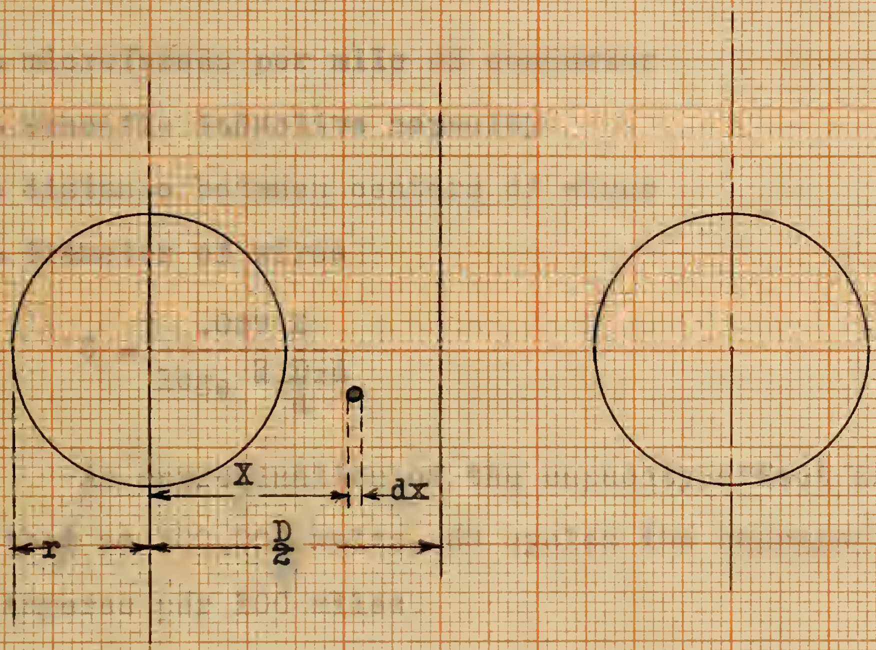

Fig 34

Capacity between parallel
Cylindrical conductors.

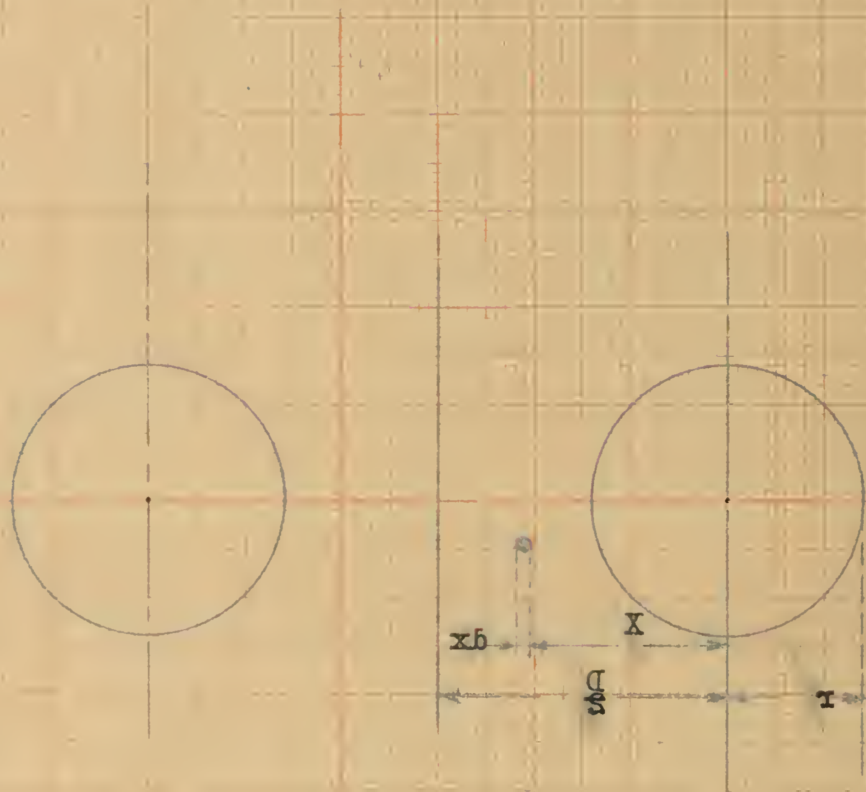

Fig 34

Capacity between parallel
Cylindrical conductors.

$$C = \frac{Q}{2\ Q\ \log_e \frac{D - r}{r}}$$

$$C = \frac{1}{2\ \log_e \frac{D - r}{r}}$$

If C = microfarads per mile of conductor

K = specific inductive capacity

D = distance between centers of wires

d = diameter of wires

$$C = \frac{.089\ K}{\log_e \frac{2\ D - d}{d}}$$

An approximation of the capacity effect is to consider that at 100,000 volts, 60 cycles the charging current is 30 amperes per 100 miles.

The Percent Method of Calculation

The ordinary calculation of machines and circuits as made by using the real values of current, voltage and impedance are of course accurate and satisfactory as to results but the quantities themselves give no real clue as to the behavior of the apparatus. Thus, for example, if it is said that a 10,000 volt 5000 K.W. generator has an armature resistance of .2 ohms it gives very little indication of what might be expected from this machine as regards regulation or I^2R loss. For this reason, and also because of simplicity of calculation, it has become customary to express resistance, reactance etc. in what is known as percent values. The method of calculation

using these percent values is known as the percent method. The percent values are always calculated with reference to the full load condition with non inductive load.

Consider the case of the generator cited in the previous paragraph, viz. 10,000 volts, 5000 K. W., armature resistance = .2 ohm. Expressed in percent it would be stated that the armature resistance is 1%, which means that the armature I R drop at full non inductive load is 1% of the normal full load voltage. The advantage of this method is readily apparent for it is seen at once from the value of % R what effect the resistance will have upon the operation of the machine. Similarly for reactance, core loss, magnetizing current etc. The full load value of current and e. m. f. are considered as 1. or 100. %.

As an illustration of the method consider the transmission line previously used in which r = 75 ohms, x = 100 ohms, receiving voltage = 20,000, power delivered = 1000 K.W., I = 50 amp. To find the generator voltage E_0.

Using the ordinary method

$$\dot{E}_0 = E + \dot{I}\,\dot{Z} = 20000 + 50(75 + j\,100)$$
$$= 20000 + 3750 + j\,5000$$
$$= 24200.$$

Using the percent method

$$E = 20000 = 1. \qquad I = 50 = 1.$$

$$I\,r = 50 \times 75 = 3750 = 18.7\ \%\ \text{of}\ 20000$$

$$I\,x = 50 \times 1000 = 5000 = 25\ \%\ \text{of}\ 20000$$

$$r = 18.7\ \% \qquad x = 25\ \%$$

$$E_o = E + I Z = 1. + 1. (.187 + j .25)$$
$$= 1. + .187 + j .25$$
$$= 1.21$$
$$E_o = 1.21 \times 20000 = 24200$$

Advantages of Percent Method of Calculation

From this it might appear that there is more work to be done in order to use the percent method but if the values of the resistance and reactance had been originally expressed in % this would not have been the case. It is seen that the regulation of the line is at once determined from the result in the latter method while in the former it must be calculated. Furthermore in the latter case the figures used as multipliers are quite frequently unity in which case the probability of error is very much reduced.

Calculations Considering Capacity

In the simple considerations of capacity effect it is sufficiently accurate to assume that the capacity is concentrated at chosen points and not distributed equally all along the line as is really the case. One method is to consider all the capacity reactance as across the lines at a point half way from the generator to the load. Another method is to consider the capacity as divided into two equal parts one of which is placed at the generating end and the other at the receiving end of the line.

Compounding Due to Capacity

Problem: Using the same transmission line as previously, viz. r = 18.75 %, x = 25 %, E of load = 1., I = 50 amp. = 1. and assuming a condenser current of 5 amperes at the load, calculate the regulation of the line.

$$I_c = 5.\ \text{amp.} = 10\ \% = .1$$
$$E_o = E + I\ Z = 1. + (1. + j\ .1)(.18.75 + j\ .25)$$
$$= 1. + .1875 - 025 + j\ (.25 + .01875)$$
$$= 1.1625 + j\ .26875$$
$$= 1.19.$$

Regulation = 19 %

Comparing this with the result on page 104 it is found that the regulation of the line is better in the case of a slight capacity load than it was when the load was non inductive. This is best illustrated by reference to the vector diagram of such a circuit. See figure 35. The voltage induced by the current flowing through the inductance of the line lags 90° behind this current. If then the current is leading the line voltage the e. m. f. induced by it will be in such phase relation to the line voltage that it will add itself thereto thus helping the voltage up or improving the regulation.

When a line has capacity load it is expected that the regulation will be better than with inductive or even non inductive load and if the capacity current is made sufficiently large the regulation may even become negative, that is the

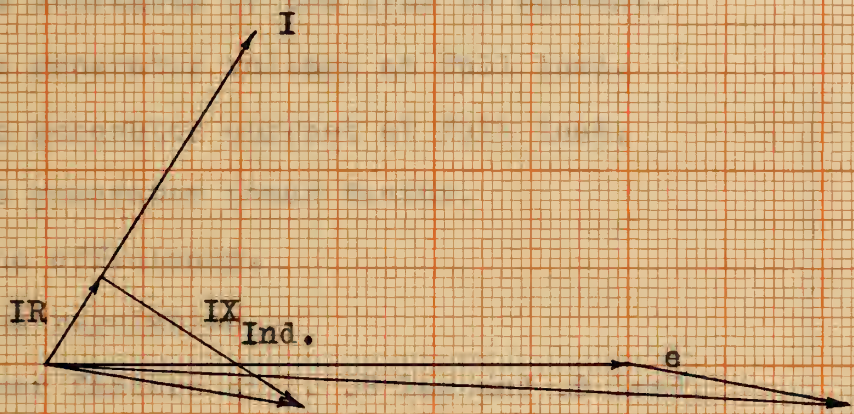

Fig. 35

Vector diagram for transmission
line having leading current.

Fig. 35

Vector diagram for transmission line having leading current.

voltage at the end of the line may be greater than the generator voltage.

This compounding effect due to leading current is often used in keeping the voltage constant for certain classes of service, the leading current being obtained by an over excited synchronous motor.

Problem: A transmission line having resistance of r = 30 ohms and x = 80 ohms delivers current to the load at 100,000 volts, 60 cycles. If the normal full load power is 10000 K. W. and the charging current is 30 amperes, find

(a) The constants of the line in percent.

(b) The generator voltage at full load.

(c) The generator current at full load.

(d) The generator power factor.

(e) Line efficiency.

(f) Line regulation

for load power factors of 1. .8 lag and .8 lead.

$$\text{Full load current} = \frac{10000000}{100000} = 100 \text{ amp.}$$

$$I\ R = 100 \times 30 = 3000 \text{ volts.}$$

$$r = \frac{3000}{100000} = 3\ \%$$

$$x = \frac{8000}{1000} = 8\ \%$$

Assuming the charging current to be taken in two parts one half at each end of the line.

$$I_c = 15 \text{ amp.} = 15\ \% \text{ at each end of line.}$$

For .8 P.F.

$$i = 100.\ \text{amp.}\ I = \frac{100.}{.8} = 125.\ \text{Amp.}$$

$$i' = 75 \text{ amp.} = 75\ \%.$$

Tabulation

P. F.	.8 Lead	1.	.8 Lag
Resistance	.03	.03	.03
Reactance	.08	.08	.08
i '= Charging current	+.15	+.15	+ 15
i	1.	1.	1.
i' load	+.75	0	-.75
i' line	+.90	+.15	-.60
$\dot{I}\ \dot{Z}$ = line drop	.0422 + j .107	.016 + j .084	.078 + j .619
$E_o = e + \dot{I}\ \dot{Z}$	.9578 + .1071j	1.018 + j0845	1.078+j .0619
E_o Real Value	96500	102200	108000
I_o Gen. Current	1. + j 1.05	1 + j .3	1. + j .45
Regulation	-3.5%	2.2%	8.%
P_o generator power	1.07	1.046	1.1
P. F. at load	.77	.98	.93
Eff. of line	.93	.96	.91
Generator P. F.	.72	.95	.84

CONCLUSION

By the use of the principles set forth in this thesis calculations of the behavior of the various electrical machines becomes readily possible. The transformer, alternator, induction motor etc. may all be treated in the same general manner as a circuit containing resistance and reactance.

Specific examples of these calculations have not been taken up in this work but may be found in various texts dealing with alternating current machines.